

Pointwise Convergence of Schrödinger Solutions Along Highly Tangential Curves

Javier Minguillón

Universidad Autónoma de Madrid

EARCO 2026, Universidad de La Rioja
May 29, 2026

Joint work with: Fernando Soria and Ana Vargas

Outline

- 1 Introduction
- 2 Tangential Problem
- 3 Main estimate at $\alpha < 1/2$
- 4 Proof
- 5 Summary

The Linear Schrödinger Equation

Fix $s > 0$ and $n \in \mathbb{N}$. Let $f \in H^s(\mathbb{R}^n)$. Consider

$$\begin{cases} i\partial_t u(x, t) + \Delta_x u(x, t) = 0, & (x, t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x, 0) = f(x), & x \in \mathbb{R}^n. \end{cases}$$

The solution is given by

$$u(x, t) = e^{it\Delta} f(x) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} e^{2\pi i t |\xi|^2} \hat{f}(\xi) d\xi.$$

The Linear Schrödinger Equation

Fix $s > 0$ and $n \in \mathbb{N}$. Let $f \in H^s(\mathbb{R}^n)$. Consider

$$\begin{cases} i\partial_t u(x, t) + \Delta_x u(x, t) = 0, & (x, t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x, 0) = f(x), & x \in \mathbb{R}^n. \end{cases}$$

The solution is given by

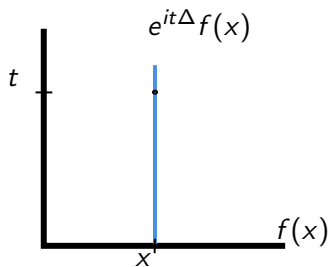
$$u(x, t) = e^{it\Delta} f(x) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} e^{2\pi i t |\xi|^2} \hat{f}(\xi) d\xi.$$

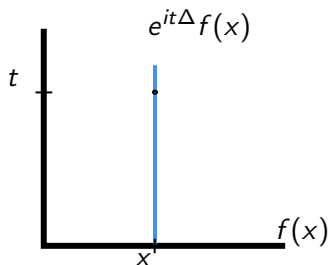
Pointwise Almost Everywhere Convergence Problem

Is

$$\lim_{t \rightarrow 0} e^{it\Delta} f(x) = f(x) \text{ a.e. } x \in \mathbb{R}^n \quad (1)$$

for all $f \in H^s(\mathbb{R}^n)$?

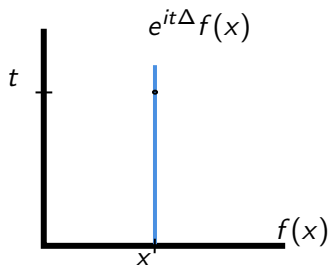
Dimension $n = 1$ 

Dimension $n = 1$ 

Theorem (Carleson, 1980)

If $f \in H^s(\mathbb{R})$ with $s \geq \frac{1}{4}$, then

$$\lim_{t \rightarrow 0} e^{it\Delta} f(x) = f(x) \text{ a.e. } x \in \mathbb{R}.$$

Dimension $n = 1$ 

Theorem (Carleson, 1980)

If $f \in H^s(\mathbb{R})$ with $s \geq \frac{1}{4}$, then

$$\lim_{t \rightarrow 0} e^{it\Delta} f(x) = f(x) \text{ a.e. } x \in \mathbb{R}.$$

The range is sharp (Dahlberg–Kenig, 1982).

Higher Dimensions: First Developments

Theorem (Sjölin, 1987; Vega, 1988)

For $n \geq 2$, $s > \frac{1}{2}$ implies a.e. convergence (1).

Theorem (Bourgain, 1991)

Let $n = 2$. There exists $\sigma < \frac{1}{2}$ such that, for any $s > \sigma$, (1) holds.

Theorem (Moyua, Vargas, Vega (1996))

Let $n = 2$. There exists $\sigma < \frac{41}{84}$ such that, for any $s > \sigma$, (1) holds.

Higher Dimensions: First Developments

Theorem (Sjölin, 1987; Vega, 1988)

For $n \geq 2$, $s > \frac{1}{2}$ implies a.e. convergence (1).

Theorem (Bourgain, 1991)

Let $n = 2$. There exists $\sigma < \frac{1}{2}$ such that, for any $s > \sigma$, (1) holds.

Theorem (Moyua, Vargas, Vega (1996))

Let $n = 2$. There exists $\sigma < \frac{41}{84}$ such that, for any $s > \sigma$, (1) holds.

Theorem (S.Lee, 2006)

For $n = 2$, $s > \frac{3}{8}$ implies convergence (1).

Higher Dimensions: Final Threshold

Theorem (Bourgain, 2016)

For $n \geq 2$, if $s < \frac{n}{2(n+1)}$, a.e. convergence (1) fails.

Higher Dimensions: Final Threshold

Theorem (Bourgain, 2016)

For $n \geq 2$, if $s < \frac{n}{2(n+1)}$, a.e. convergence (1) fails.

Theorem (Du–Guth–Li (2017))

For $n = 2$, if $s > \frac{1}{3}$, then (1) holds.

Higher Dimensions: Final Threshold

Theorem (Bourgain, 2016)

For $n \geq 2$, if $s < \frac{n}{2(n+1)}$, a.e. convergence (1) fails.

Theorem (Du–Guth–Li (2017))

For $n = 2$, if $s > \frac{1}{3}$, then (1) holds.

Theorem (Du–Zhang (2019))

Given $n \geq 2$, if $s > \frac{n}{2(n+1)}$, then (1) holds.

Higher Dimensions: Final Threshold

Theorem (Bourgain, 2016)

For $n \geq 2$, if $s < \frac{n}{2(n+1)}$, a.e. convergence (1) fails.

Theorem (Du–Guth–Li (2017))

For $n = 2$, if $s > \frac{1}{3}$, then (1) holds.

Theorem (Du–Zhang (2019))

Given $n \geq 2$, if $s > \frac{n}{2(n+1)}$, then (1) holds.

Thus, the sharp threshold for a.e. convergence to the initial datum is

$$s(n) = \frac{n}{2(n+1)}.$$

Convergence at this endpoint remains an open question.

Tangential problem

Fix $0 < \alpha \leq 1$ and $C > 0$. Consider curves $\gamma : [0, 1] \rightarrow \mathbb{R}^n$ that satisfy $\gamma(0) = 0$ and, for arbitrary $t, t' \in \mathbb{R}$,

$$|\gamma(t) - \gamma(t')| \leq C|t - t'|^\alpha.$$

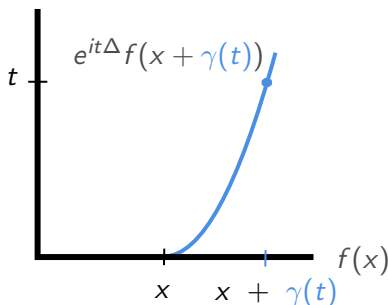
Denote by Γ_C^α the family containing such curves γ .

Tangential problem

Fix $0 < \alpha \leq 1$ and $C > 0$. Consider curves $\gamma : [0, 1] \rightarrow \mathbb{R}^n$ that satisfy $\gamma(0) = 0$ and, for arbitrary $t, t' \in \mathbb{R}$,

$$|\gamma(t) - \gamma(t')| \leq C|t - t'|^\alpha.$$

Denote by Γ_C^α the family containing such curves γ .



We may assume $C = 1$.

Developments: $n = 1$

Tangential convergence problem:

Does

$$\lim_{t \rightarrow 0} e^{it\Delta} f(x + \gamma(t)) = f(x) \quad \text{a.e. } x \in \mathbb{R}^n \quad (2)$$

hold for all $\gamma \in \Gamma_1^\alpha$ and all $f \in H^s(\mathbb{R}^n)$?

The problem was solved in dimension $n = 1$ up to the endpoint.

Developments: $n = 1$

Tangential convergence problem:

Does

$$\lim_{t \rightarrow 0} e^{it\Delta} f(x + \gamma(t)) = f(x) \quad \text{a.e. } x \in \mathbb{R}^n \quad (2)$$

hold for all $\gamma \in \Gamma_1^\alpha$ and all $f \in H^s(\mathbb{R}^n)$?

The problem was solved in dimension $n = 1$ up to the endpoint.

Theorem (Cho, Lee, Vargas (2012))

Let $n = 1$, $0 < \alpha \leq 1$ and $s > \max \{ \frac{1}{2} - \alpha, \frac{1}{4} \}$. Then (2) holds.

Developments: $n = 1$

Tangential convergence problem:

Does

$$\lim_{t \rightarrow 0} e^{it\Delta} f(x + \gamma(t)) = f(x) \quad \text{a.e. } x \in \mathbb{R}^n \quad (2)$$

hold for all $\gamma \in \Gamma_1^\alpha$ and all $f \in H^s(\mathbb{R}^n)$?

The problem was solved in dimension $n = 1$ up to the endpoint.

Theorem (Cho, Lee, Vargas (2012))

Let $n = 1$, $0 < \alpha \leq 1$ and $s > \max \left\{ \frac{1}{2} - \alpha, \frac{1}{4} \right\}$. Then (2) holds.

- They also show that $s \geq \max \left\{ \frac{1}{2} - \alpha, \frac{1}{4} \right\}$ is necessary.

Developments: $n = 1$

Tangential convergence problem:

Does

$$\lim_{t \rightarrow 0} e^{it\Delta} f(x + \gamma(t)) = f(x) \quad \text{a.e. } x \in \mathbb{R}^n \quad (2)$$

hold for all $\gamma \in \Gamma_1^\alpha$ and all $f \in H^s(\mathbb{R}^n)$?

The problem was solved in dimension $n = 1$ up to the endpoint.

Theorem (Cho, Lee, Vargas (2012))

Let $n = 1$, $0 < \alpha \leq 1$ and $s > \max \left\{ \frac{1}{2} - \alpha, \frac{1}{4} \right\}$. Then (2) holds.

- They also show that $s \geq \max \left\{ \frac{1}{2} - \alpha, \frac{1}{4} \right\}$ is necessary.
- Ding and Niu proved in 2017 that, if $\frac{1}{2} \leq \alpha \leq 1$ and $s \geq \frac{1}{4}$, then (2), holds.

Developments: $n \geq 2$

Problem:

Does

$$\lim_{t \rightarrow 0} e^{it\Delta} f(x + \gamma(t)) = f(x) \quad \text{a.e. } x \in \mathbb{R}^n \quad (2)$$

hold for all $\gamma \in \Gamma_1^\alpha$ and all $f \in H^s(\mathbb{R}^n)$?

Theorem (Li, Wang (2021))

Fix $n = 2$ and $\frac{1}{2} \leq \alpha \leq 1$. Let $s > \frac{3}{8}$. If $f \in H^s(\mathbb{R}^2)$, then (2) holds.

Developments: $n \geq 2$

Problem:

Does

$$\lim_{t \rightarrow 0} e^{it\Delta} f(x + \gamma(t)) = f(x) \quad \text{a.e. } x \in \mathbb{R}^n \quad (2)$$

hold for all $\gamma \in \Gamma_1^\alpha$ and all $f \in H^s(\mathbb{R}^n)$?

Theorem (Li, Wang (2021))

Fix $n = 2$ and $\frac{1}{2} \leq \alpha \leq 1$. Let $s > \frac{3}{8}$. If $f \in H^s(\mathbb{R}^2)$, then (2) holds.

Theorem (Cao, Miao (2023); M. (2024))

Fix $n \geq 2$ and $\frac{1}{2} \leq \alpha \leq 1$. Let $s > \frac{n}{2(n+1)}$. If $f \in H^s(\mathbb{R}^n)$, then (2) holds.

Developments: $n \geq 2$

Problem:

Does

$$\lim_{t \rightarrow 0} e^{it\Delta} f(x + \gamma(t)) = f(x) \quad \text{a.e. } x \in \mathbb{R}^n \quad (2)$$

hold for all $\gamma \in \Gamma_1^\alpha$ and all $f \in H^s(\mathbb{R}^n)$?

Theorem (Li, Wang (2021))

Fix $n = 2$ and $\frac{1}{2} \leq \alpha \leq 1$. Let $s > \frac{3}{8}$. If $f \in H^s(\mathbb{R}^2)$, then (2) holds.

Theorem (Cao, Miao (2023); M. (2024))

Fix $n \geq 2$ and $\frac{1}{2} \leq \alpha \leq 1$. Let $s > \frac{n}{2(n+1)}$. If $f \in H^s(\mathbb{R}^n)$, then (2) holds.

What about $\alpha < \frac{1}{2}$?

Main Result

Consider the model curve

$$\gamma_0(t) = (t^{\alpha_1}, \dots, t^{\alpha_n}), \quad 0 < \alpha = \min_j \alpha_j < \frac{1}{2}.$$

Theorem (M–Soria–Vargas, 2026)

Let

$$s_0(n, \alpha) = \max \left\{ \frac{1 - 2\alpha}{2}, \frac{n}{2(n+1)} \right\}.$$

Then for every $s > s_0(n, \alpha)$,

$$\lim_{t \rightarrow 0} e^{it\Delta} f(x + \gamma_0(t)) = f(x) \quad \text{a.e. } x \in \mathbb{R}^n, \quad (3)$$

for all $f \in H^s(\mathbb{R}^n)$.

The condition is sharp up to the endpoint.

The associated maximal estimate

Convergence (3) is a consequence of the following.

Theorem

Let $n \geq 1$. Fix $0 \leq \alpha \leq 1$. For any $\varepsilon > 0$, there exists a positive constant C_ε such that,

$$\left\| \sup_{0 < t < 1} \left| e^{it\Delta} f(x + \gamma_0(t)) \right| \right\|_{L^2(B_1^n(0))} \leq C_\varepsilon \|f\|_{H^{s_0(n,\alpha)+\varepsilon}(\mathbb{R}^n)}, \quad (4)$$

holds for all $f \in H^{s_0(n,\alpha)+\varepsilon}(\mathbb{R}^n)$.

Reductions

By Littlewood-Paley decomposition, the previous maximal estimate follows from proving

$$\left\| \sup_{0 < t < 1} \left| e^{it\Delta} f(x + \gamma_0(t)) \right| \right\|_{L^2(B_1^n(0))} \leq C_\varepsilon R^{s_0(n, \alpha) + \varepsilon} \|f\|_2,$$

for all f with $\text{supp } \widehat{f} \subset \{\xi \in \mathbb{R}^n : |\xi| \sim R\}$ and all $R \gg 1$.

By Parabolic rescaling, $x \rightarrow \frac{x'}{R}$, $t \rightarrow \frac{t'}{R^2}$ the above is equivalent to

$$\left\| \sup_{0 < t \leq R^2} \left| e^{it\Delta} f\left(x + R\gamma_0\left(\frac{t}{R^2}\right)\right) \right| \right\|_{L^2(B_R^n(0))} \leq C_\varepsilon R^{s_0(n, \alpha) + \varepsilon} \|f\|_2,$$

for all f with $\text{supp } \widehat{f} \subset \{\xi \in \mathbb{R}^n : |\xi| \sim 1\}$.

Reductions II

Last reduction

$$\left\| \sup_{0 < t \leq R^2} \left| e^{it\Delta} f \left(x + R\gamma_0 \left(\frac{t}{R^2} \right) \right) \right| \right\|_{L^2(B_R^n(0))} \leq C_\varepsilon R^{\frac{n}{2(n+1)} + \varepsilon} \|f\|_2,$$

By the Time Localization Lemma, we only need to prove the below.

Reductions II

Last reduction

$$\left\| \sup_{0 < t \leq R^2} \left| e^{it\Delta} f \left(x + R\gamma_0 \left(\frac{t}{R^2} \right) \right) \right| \right\|_{L^2(B_R^n(0))} \leq C_\varepsilon R^{\frac{n}{2(n+1)} + \varepsilon} \|f\|_2,$$

By the Time Localization Lemma, we only need to prove the below.

Theorem

Let $n \geq 1$ and $0 \leq \alpha \leq 1$. For any $\varepsilon > 0$, there exists a constant C_ε such that, for all $R \gg 1$,

$$\left\| \sup_{0 < t \leq R} \left| e^{it\Delta} f \left(x + R\gamma_0 \left(\frac{t}{R^2} \right) \right) \right| \right\|_{L^2(B_R^n(0))} \leq C_\varepsilon R^{s_0(n,\alpha) + \varepsilon} \|f\|_2.$$

holds for all f with $\text{supp } \hat{f} \subset A(1) = \{\xi \in \mathbb{R}^n : |\xi| \sim 1\}$.

Reduced Maximal Estimate

As we have seen, the main theorem follows from the following.

Key Maximal Estimate

For all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that, for all $R \gg 1$, and all f with $\text{supp } \hat{f} \subset B_1^n(0)$,

$$\left\| \sup_{0 < t < R} |e^{it\Delta} f(x + \theta(t))| \right\|_{L^2(B_R(0))} \leq C_\varepsilon R^{s_0(n, \alpha) + \varepsilon} \|f\|_2,$$

where $\theta(t) = R\gamma_0 \left(\frac{t}{R^2} \right) = (R^{1-2\alpha_1} t^{\alpha_1}, \dots, R^{1-2\alpha_n} t^{\alpha_n})$.

We split the supremum into $t \in (0, 1)$ (small times) and $t \in (1, R)$ (large times).

Small Times: Local Constancy of $e^{it\Delta}f$

Local Constancy Property

Suppose $\text{supp } \widehat{f} \subset B_1^n(0)$. Let $|x - y| \leq 1$ and $|t - s| \leq 1$. Then

$$\left| e^{it\Delta} f(x) \right| \sim \left| e^{is\Delta} f(y) \right|.$$

Therefore

$$\sup_{0 < t < 1} \left| e^{it\Delta} f(x + \theta(t)) \right| \lesssim \max_{k \in \mathcal{N}(\theta)} \left(\int_0^1 \int_{B_1(x+k)} \left| e^{is\Delta} f(y) \right|^2 dy ds \right)^{1/2},$$

where $\mathcal{N}(\theta)$ is the set of integer points in the 1-neighborhood of $\theta([0, 1])$.

Small Times: Geometric Length of the Curve

Key computation. Recall $\theta(t) = (R^{1-2\alpha_1} t^{\alpha_1}, \dots, R^{1-2\alpha_n} t^{\alpha_n})$.

Its derivative satisfies, with $\alpha = \min_j \alpha_j$,

$$|\theta'(t)| \simeq R^{1-2\alpha} t^{\alpha-1}.$$

Small Times: Geometric Length of the Curve

Key computation. Recall $\theta(t) = (R^{1-2\alpha_1} t^{\alpha_1}, \dots, R^{1-2\alpha_n} t^{\alpha_n})$.

Its derivative satisfies, with $\alpha = \min_j \alpha_j$,

$$|\theta'(t)| \simeq R^{1-2\alpha} t^{\alpha-1}.$$

Therefore,

$$\text{length}(\theta[0, 1]) = \int_0^1 |\theta'(t)| dt \simeq_\alpha R^{1-2\alpha}.$$

We conclude that

$$\#\mathcal{N}(\theta) \lesssim R^{1-2\alpha}.$$

Small Times: Conclusion

Squaring and integrating over $x \in B_R(0)$, and using local constancy,

$$\begin{aligned}
 & \left\| \sup_{0 < t < 1} \left| e^{it\Delta} f(x + \theta(t)) \right| \right\|_{L^2(B_R(0))}^2 \\
 & \lesssim \int_{B_R(0)} \sum_{k \in \mathcal{N}(\theta)} \int_0^1 \int_{B_1(x+k)} \left| e^{is\Delta} f(y) \right|^2 dy ds dx. \\
 & \lesssim \int_0^1 \int_{\mathbb{R}^n} |e^{is\Delta} f(y)|^2 \left(\sum_{k \in \mathcal{N}(\theta)} \underbrace{\int_{B_R(0)} \chi_{B_1(x+k)}(y) dx}_{=c_n} \right) dy ds. \\
 & \simeq \#\mathcal{N}(\theta) \int_0^1 \|e^{is\Delta} f\|_2^2 ds \\
 & \lesssim R^{1-2\alpha} \|f\|_2^2
 \end{aligned}$$

Large Times: The Set $X_{\lambda,\eta}$

For each $j \in \mathbb{Z}^n \cap B_R^n(0)$, pick $t_j \in [1, R]$ such that

$$\sup_{1 < t < R} |e^{it\Delta} f(j + \theta(t))| \leq 2 \left| e^{it_j\Delta} f(j + \theta(t_j)) \right|.$$

Let Q_j be the unit lattice cube containing $(j + \theta(t_j), t_j)$.

Define $X = \bigcup_{|j| \leq R} Q_j$.

Large Times: The Set $X_{\lambda,\eta}$

For each $j \in \mathbb{Z}^n \cap B_R^n(0)$, pick $t_j \in [1, R]$ such that

$$\sup_{1 < t < R} |e^{it\Delta} f(j + \theta(t))| \leq 2 \left| e^{it_j\Delta} f(j + \theta(t_j)) \right|.$$

Let Q_j be the unit lattice cube containing $(j + \theta(t_j), t_j)$.

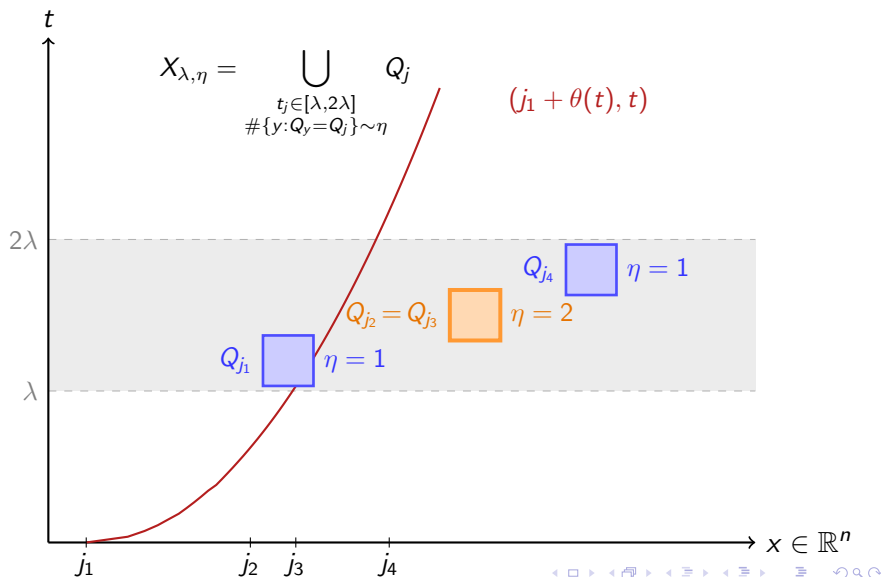
Define $X = \bigcup_{|j| \leq R} Q_j$.

Decompose X by dyadic **time scale** $\lambda \in [1, R]$ and dyadic **multiplicity** $\eta \geq 1$.

$$X_{\lambda,\eta} := \bigcup \{ Q_j : t_j \in [\lambda, 2\lambda], \# \{ y \in B_R^n(0) \cap \mathbb{Z}^n : Q_y = Q_j \} \simeq \eta \}.$$

One can prove $\eta \lesssim \max\{1, R^{1-2\alpha} \lambda^{\alpha-1}\}$

Large Times: The Set $X_{\lambda,\eta}$



Large Times: The Du–Zhang Estimate

Du–Zhang (Ann. of Math., 2019)

For $X \subset B_\lambda^{n+1}(0)$ a union of unit cubes. Define a density

$$\varphi_X := \max_{\substack{B_r^{n+1}(z) \subset B_\lambda^{n+1}(0) \\ z \in \mathbb{R}^{n+1}, r \geq 1}} \frac{\#\{Q_k \subset X : Q_k \subset B_r^{n+1}(z)\}}{r^n},$$

then

$$\left\| e^{it\Delta} f \right\|_{L^2(X)} \leq C_\varepsilon \varphi_X^{\frac{1}{n+1}} \lambda^{\frac{n}{2(n+1)} + \varepsilon} \|f\|_2.$$

Large Times: Density Factor and Its Bound

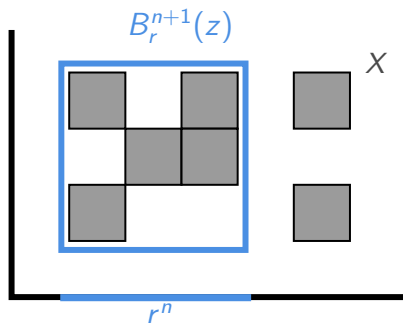
The method relies on measuring a density of unit cubes defined as

$$\varphi_X := \max_{\substack{B_r^{n+1}(z) \subset B_R^{n+1}(0) \\ z \in \mathbb{R}^{n+1}, r \geq 1}} \frac{\#\{Q_k \subset X : Q_k \subset B_r^{n+1}(z)\}}{r^n}.$$

Large Times: Density Factor and Its Bound

The method relies on measuring a density of unit cubes defined as

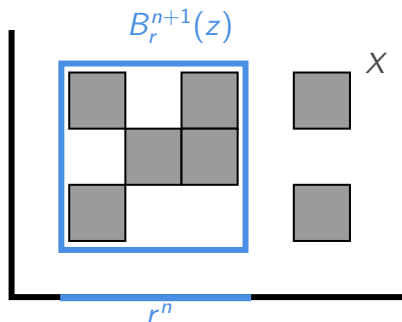
$$\varphi_X := \max_{\substack{B_r^{n+1}(z) \subset B_R^{n+1}(0) \\ z \in \mathbb{R}^{n+1}, r \geq 1}} \frac{\#\{Q_k \subset X : Q_k \subset B_r^{n+1}(z)\}}{r^n}.$$



Large Times: Density Factor and Its Bound

The method relies on measuring a density of unit cubes defined as

$$\varphi_X := \max_{\substack{B_r^{n+1}(z) \subset B_R^{n+1}(0) \\ z \in \mathbb{R}^{n+1}, r \geq 1}} \frac{\#\{Q_k \subset X : Q_k \subset B_r^{n+1}(z)\}}{r^n}.$$



Key Lemma

$$\varphi_{X_{\lambda,\eta}} \lesssim \eta^{-1} \max\{1, R^{1-2\alpha} \lambda^{\alpha-1}\}$$

This lemma is proven looking at the length of $\theta[\lambda, 2\lambda]$ and exploiting the multiplicity η .

Large Times: The Rogers Estimate

For each lattice cube center $k \in \lambda\mathbb{Z}^n$, the above gives

$$\begin{aligned} \eta^{\frac{1}{2}} \left\| e^{it\Delta} f \right\|_{L^2(X_{\lambda,\eta} \cap (B_{\lambda}^n(k) \times [0, 2\lambda]))} &\leq C_{\varepsilon} \eta^{\frac{1}{2}} \varphi_{X_{\lambda,\eta}}^{\frac{1}{n+1}} \lambda^{\frac{n}{2(n+1)} + \varepsilon} \|f\|_2 \\ &\leq C_{\varepsilon} \eta^{\frac{1}{2}} (\eta^{-1} \max\{1, R^{1-2\alpha} \lambda^{\alpha-1}\})^{\frac{1}{n+1}} \lambda^{\frac{n}{2(n+1)} + \varepsilon} \|f\|_2 \leq C_{\varepsilon} R^{s_0(n,\alpha) + \varepsilon} \|f\|_2 \end{aligned}$$

The last step is proven via the bound $\eta \lesssim \max\{1, R^{1-2\alpha} \lambda^{\alpha-1}\}$ and an optimization in both λ and α .

Rogers (Adv. Math., 2008)

A $(2, 2)$ -strong estimate over $B_{\lambda}^n(k) \times [0, \lambda]$ uniform over all $k \in \lambda\mathbb{Z}^n$, implies an estimate over $\mathbb{R}^n \times [0, \lambda]$ with a λ^{ε} loss.

By the above,

$$\left\| e^{it\Delta} f \right\|_{L^2(X_{\lambda,\eta})} \lesssim C_{\varepsilon} R^{\varepsilon} \eta^{-\frac{1}{2}} R^{s_0(n,\alpha) + \varepsilon} \|f\|_2$$

Large Times: Conclusion

By local constancy,

$$\begin{aligned} & \left\| \sup_{1 < t < R} |e^{it\Delta} f(x + \theta(t))| \right\|_{L^2(B_R(0))}^2 \lesssim \sum_{|j| \leq R} \sup_{1 < t < R} |e^{it\Delta} f(j + \theta(t))|^2 \\ & \lesssim \sum_{|j| \leq R} |e^{it_j \Delta} f(j + \theta(t_j))|^2 \lesssim \sum_{\lambda, \eta} \eta \cdot \|e^{it\Delta} f\|_{L^2(X_{\lambda, \eta})}^2. \end{aligned}$$

Applying the estimate $\|e^{it\Delta} f\|_{L^2(X_{\lambda, \eta})} \lesssim C_\varepsilon R^\varepsilon \eta^{-1/2} R^{s_0(n, \alpha)} \|f\|_2$,

$$\lesssim \sum_{\lambda, \eta} C_\varepsilon R^{2s_0(n, \alpha) + 2\varepsilon} \|f\|_2^2 \leq C_\varepsilon R^{2s_0(n, \alpha) + 3\varepsilon} \|f\|_2^2$$

Summary and Remarks

Main Result

For the model curve $\gamma_0(t) = (t^{\alpha_1}, \dots, t^{\alpha_n})$ with $\alpha = \min_j \alpha_j < \frac{1}{2}$, the critical Sobolev exponent for a.e. convergence is

$$s_0(n, \alpha) = \max \left\{ \frac{1 - 2\alpha}{2}, \frac{n}{2(n+1)} \right\}.$$

Remarks:

- As $\alpha \nearrow \frac{1}{2}$ the threshold $s_0(n, \alpha) \rightarrow \frac{n}{2(n+1)}$, recovering the result of Cao–Miao (2023).
- As $\alpha \searrow 0$ the curve becomes more and more tangential and the required regularity increases towards $s = \frac{1}{2}$.
- The proof combines: **(1)** stability of $e^{it\Delta}$, **(2)** the Du–Zhang L^2 estimate, and **(3)** a geometric density analysis of the curve.

References I



J. Bourgain, *A note on the Schrödinger maximal function*, J. Anal. Math. **130** (2016), 393–396.



C. Cho, S. Lee, and A. Vargas, *Problems on pointwise convergence of solutions to the Schrödinger equation*, J. Fourier Anal. Appl. **18** (2012), 972–994.



X. Du and R. Zhang, *Sharp L^2 estimates of the Schrödinger maximal function in high dimensions*, Ann. of Math. **189** (2019), 837–861.



J. Minguión, F. Soria, A. Vargas, *On the pointwise convergence of solutions to the Schrödinger equation along certain highly tangential curves*, arXiv:2604.12795 (2026).



K. M. Rogers, *A local smoothing estimate for the Schrödinger equation*, Adv. Math. **219** (2008), no. 6, 2105–2122.

Self-improving properties of Poincaré-type inequalities

Alejandro Claros

BCAM – Basque Center for Applied Mathematics
UPV/EHU

This is part of a joint work with Carlos Pérez (BCAM, UPV/EHU)
and Linfei Zheng (Tianjin University).

EARCO 2026

Plan for the talk

- 1 Classical and weighted Poincaré-Sobolev inequalities.
- 2 Generalized Poincaré inequalities and self-improvement.
- 3 The new A_r self-improving theorem.
- 4 Applications and extensions.

Introduction and motivation

Theorem ((1, 1) Poincaré inequality)

There exists a dimensional constant $c_n > 0$ such that, for every cube $Q \subset \mathbb{R}^n$ and every $f \in W^{1,1}(Q)$,

$$\frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx \leq c_n \ell(Q) \frac{1}{|Q|} \int_Q |\nabla f(x)| \, dx,$$

where $f_Q = \frac{1}{|Q|} \int_Q f(y) dy$.

- Subrepresentation formula:

$$|f(x) - f_Q| \leq c_n I_1(|\nabla f| \chi_Q)(x), \quad x \in Q.$$

$$\text{where } I_\alpha g(x) = \int_{\mathbb{R}^n} \frac{g(y)}{|x - y|^{n-\alpha}} \, dy.$$

Poincaré inequalities

The (1, 1) Poincaré inequality implies:

- (1, p) Poincaré inequality: for $1 \leq p < \infty$,

$$\frac{1}{|Q|} \int_Q |f - f_Q| \, dx \lesssim \ell(Q) \left(\frac{1}{|Q|} \int_Q |\nabla f|^p \, dx \right)^{\frac{1}{p}}.$$

- (p , p) Poincaré inequality: for $1 \leq p < \infty$,

$$\left(\frac{1}{|Q|} \int_Q |f - f_Q|^p \, dx \right)^{\frac{1}{p}} \lesssim \ell(Q) \left(\frac{1}{|Q|} \int_Q |\nabla f|^p \, dx \right)^{\frac{1}{p}}.$$

- Poincaré-Sobolev inequality: for $1 \leq p < n$,

$$\left(\frac{1}{|Q|} \int_Q |f - f_Q|^{p^*} \, dx \right)^{\frac{1}{p^*}} \lesssim \ell(Q) \left(\frac{1}{|Q|} \int_Q |\nabla f|^p \, dx \right)^{\frac{1}{p}}, \quad \frac{1}{p} - \frac{1}{p^*} = \frac{1}{n}.$$

Motivation: Elliptic operators

$$L = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots \frac{\partial^2}{\partial x_n^2} = \operatorname{div}(\nabla), \quad Lu = 0.$$

Elliptic operators:

$$Lu = \operatorname{div}(A(x) \cdot \nabla u) = 0,$$

where $A(x)$ is a $n \times n$ matrix such that

$$c|\xi|^2 \leq A(x)\xi \cdot \xi \leq C|\xi|^2.$$

Goal: To prove local Hölder continuity of the solutions of the equation.

- Di Giorgi (1957), Nash (1957).
- Moser (1960) (Moser iteration method)
 - (2,2) Poincaré inequality.
 - Poincaré-Sobolev inequality.

$$\left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^{2^*} dx \right)^{\frac{1}{2^*}} \leq c_n \ell(Q) \left(\frac{1}{|Q|} \int_Q |\nabla f(x)|^2 dx \right)^{\frac{1}{2}}.$$

Motivation: Degenerate elliptic operators

$$Lu = \operatorname{div}(A(x) \cdot \nabla u) = 0,$$

$$c|\xi|^2 w(x) \leq A(x)\xi \cdot \xi \leq C|\xi|^2 w(x)$$

where w is a weight.

Goal: To prove local Hölder continuity of the solutions of the equation.

Moser iteration method: Fabes, Kenig, Serapioni (1982).

Let $w \in A_2$,

- (2,2) Poincaré inequality

$$\left(\frac{1}{w(Q)} \int_Q |f - f_Q|^2 w dx \right)^{\frac{1}{2}} \leq c_w \ell(Q) \left(\frac{1}{w(Q)} \int_Q |\nabla f|^2 w dx \right)^{\frac{1}{2}}.$$

- Poincaré-Sobolev inequality, for some $2_w^* > 2$:

$$\left(\frac{1}{w(Q)} \int_Q |f - f_Q|^{2_w^*} w dx \right)^{\frac{1}{2_w^*}} \leq C_w \ell(Q) \left(\frac{1}{w(Q)} \int_Q |\nabla f|^2 w dx \right)^{\frac{1}{2}}.$$

Generalized Poincaré inequalities

Self-improving of Poincaré-type inequalities

Generalized Poincaré inequality

$$\frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx \leq a(Q), \quad Q \in \mathcal{Q}.$$

Here $\mathcal{Q}$ denotes the family of all cubes and

$$a : \mathcal{Q} \longrightarrow (0, \infty)$$

is a functional on cubes. The problem becomes:

Self-improvement problem

Which geometric conditions on a imply a self-improvement of the LHS with respect to a measure $w(x) \, dx$?

This point of view goes back to Franchi, Pérez and Wheeden 1998.

The model example

Let $w \in A_p$. From the $(1, 1)$ Poincaré inequality and the A_p condition,

$$\begin{aligned} \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx &\leq c_n \ell(Q) \frac{1}{|Q|} \int_Q |\nabla f(x)| dx \\ &\leq c_n [w]_{A_p}^{\frac{1}{p}} \ell(Q) \left(\frac{1}{w(Q)} \int_Q |\nabla f|^p w dx \right)^{\frac{1}{p}} \\ &:= a_f(Q). \end{aligned}$$

More generally, we keep in mind functionals of the form

$$a(Q) = \ell(Q)^\alpha \left(\frac{\mu(Q)}{w(Q)} \right)^{\frac{1}{p}},$$

where μ is a locally finite measure and $\alpha, p > 0$. The previous example corresponds to $\alpha = 1$ and $d\mu = |\nabla f|^p w dx$.

Self-improving of Poincaré-type inequalities

Definition

We say that $a \in D_p(w)$ if, for every cube Q and every pairwise disjoint family $\{Q_j\}_j \subset Q$,

$$\sum_j a(Q_j)^p w(Q_j) \leq C^p a(Q)^p w(Q).$$

- This condition was introduced by Franchi, Pérez and Wheeden in the setting of spaces of homogeneous type.
- The best constant is denoted by $\|a\|_{D_p(w)}$, and $\|a\|_{D_p(w)} \geq 1$.
- For $a(Q) = \ell(Q)^\alpha \left(\frac{\mu(Q)}{w(Q)} \right)^{\frac{1}{p}}$,

$$\sum_j a(Q_j)^p w(Q_j) = \sum_j \ell(Q_j)^{\alpha p} \mu(Q_j) \leq \ell(Q)^{\alpha p} \mu(Q) = a(Q)^p w(Q).$$

Hence $a \in D_p(w)$ with constant $\|a\|_{D_p(w)} = 1$.

Self-improving of Poincaré-type inequalities with $D_p(w)$

Franchi, Pérez and Wheeden proved this self-improvement in the general setting of spaces of homogeneous type, with exponential dependence on the A_∞ constant.

In the Euclidean setting with cubes, Canto and Pérez later improved this to linear dependence on $[w]_{A_\infty}$.

Theorem (Franchi, Pérez, Wheeden 98; Canto, Pérez 21)

Let $w \in A_\infty$ and let $a \in D_p(w)$. Let $f \in L^1_{\text{loc}}$ be such that

$$\frac{1}{|Q|} \int_Q |f(x) - f_Q| dx \leq a(Q)$$

for every cube Q . Then, for every cube Q ,

$$\|f - f_Q\|_{L^{p,\infty}(Q, \frac{w dx}{w(Q)})} \lesssim_n p[w]_{A_\infty} \|a\|_{D_p(w)} a(Q).$$

Self-improving of Poincaré-type inequalities

Pérez and Rela introduced a stronger condition that preserves smallness.

Definition

We say that $a \in SD_p^s(w)$ if, for every pairwise disjoint family $\{Q_j\}_j \subset Q$,

$$\sum_j a(Q_j)^p w(Q_j) \leq \|a\|_{SD_p^s(w)}^p \left(\frac{|\cup_j Q_j|}{|Q|} \right)^{\frac{p}{s}} a(Q)^p w(Q).$$

The same model example functional satisfies this stronger property. For every pairwise disjoint family $\{Q_j\}_j \subset Q$,

$$\begin{aligned} \sum_j a(Q_j)^p w(Q_j) &= \sum_j \ell(Q_j)^{\alpha p} \mu(Q_j) \leq \left(\sup_j |Q_j| \right)^{\frac{\alpha p}{n}} \sum_j \mu(Q_j) \\ &\leq |\cup_j Q_j|^{\frac{\alpha p}{n}} \mu(Q) = \left(\frac{|\cup_j Q_j|}{|Q|} \right)^{\frac{p}{n/\alpha}} a(Q)^p w(Q). \end{aligned}$$

Hence $a \in SD_p^{n/\alpha}(w)$ with constant $\|a\|_{SD_p^{n/\alpha}(w)} = 1$.

Self-improving of Poincaré-type inequalities with $SD_p^s(w)$

Pérez and Rela proved the following self-improving result assuming $w \in A_\infty$.

Later, Lerner, Lorist and Ombrosi showed that the same conclusion is valid for arbitrary weights.

Theorem (Pérez, Rela 19; Lerner, Lorist, Ombrosi 22)

Let $a \in SD_p^s(w)$ and let $f \in L_{\text{loc}}^1$ be such that

$$\frac{1}{|Q|} \int_Q |f(x) - f_Q| dx \leq a(Q)$$

for every cube Q . Then, for every cube Q ,

$$\left(\frac{1}{w(Q)} \int_Q |f - f_Q|^p w dx \right)^{\frac{1}{p}} \lesssim_n (1+s) \|a\|_{SD_p^s(w)} a(Q).$$

A natural question

For the $SD_p^s(w)$ condition, the A_∞ assumption can be removed.
One may ask whether the same thing happens for the weaker $D_p(w)$ condition.

Question

Does the $D_p(w)$ self-improving theorem remain true for arbitrary weights?

The answer is negative. The key example is the constant functional

$$a(Q) = \|f\|_{\text{BMO}} \in D_p(w) \quad \text{for every } p \text{ and every weight } w$$

If the $D_p(w)$ -theorem were valid for arbitrary weights, we would obtain

$$\|f\|_{\text{BMO}_{w,w}} \lesssim \|f\|_{\text{BMO}}.$$

But, by the characterization of Ombrosi, Pérez, Rela and Rivera-Ríos (2020), the embedding

$$\text{BMO} \hookrightarrow \text{BMO}_{w,w}$$

holds if and only if $w \in A_\infty$. Moreover, we show that the linear dependence $[w]_{A_\infty}$ is sharp.

From Generalized Poincaré to Poincaré–Sobolev Inequalities

The $SD_p^s(w)$ theorem is valid for arbitrary weights:

$$a \in SD_p^s(w) \implies L^p(w)\text{-self-improvement.}$$

In this work, we show that if the weight satisfies an additional condition such as $w \in A_r$, then one can go beyond $L^p(w)$ and obtain a generalized Poincaré–Sobolev improvement.

Main result

Theorem (C., Pérez, Zheng)

Let $1 \leq p, r, s < \infty$, let $w \in A_r$, and assume

$$\frac{1}{|Q|} \int_Q |f(x) - f_Q| dx \leq a(Q)$$

for every cube Q . If $a \in SD_p^s(w)$ and $p < rs$, then for every cube Q ,

$$\|f - f_Q\|_{L^{p_{r,s}^*, \infty}(Q, \frac{w dx}{w(Q)})} \leq c_n p_{r,s}^*[w]_{A_\infty} [\frac{1}{w}]_{A_r}^{rs} \|a\| a(Q),$$

where

$$\frac{1}{p} - \frac{1}{p_{r,s}^*} = \frac{1}{rs}.$$

- This is a generalized Poincaré-Sobolev inequality. The parameter s plays the role of the dimension.
- The proof is based on a good-lambda argument.

For the model functional

$$a_f(Q) = c_n[w]_{A_p}^{\frac{1}{p}} \ell(Q) \left(\frac{1}{w(Q)} \int_Q |\nabla f|^p w \, dx \right)^{\frac{1}{p}} \in SD_p^n(w).$$

Thus the parameter s in the abstract theorem behaves as a dimension. Therefore the previous result gives

$$\|f - f_Q\|_{L^{p_{r,n}^*, \infty}(Q, \frac{w \, dx}{w(Q)})} \lesssim \ell(Q) \left(\frac{1}{w(Q)} \int_Q |\nabla f|^p w \, dx \right)^{\frac{1}{p}},$$

where the exponent is

$$\frac{1}{p} - \frac{1}{p_{r,n}^*} = \frac{1}{nr}.$$

This exponent is sharp for the whole class A_r . Indeed, in the range $1 \leq r \leq p$, we can prove that the corresponding estimate cannot hold for all $w \in A_r$ with $L^{q, \infty}$ on the left-hand side for any $q > p_{r,n}^*$.

Main theorem

In the critical and supercritical range $p \geq rs$, we obtain exponential integrability.

Theorem (C., Pérez, Zheng)

Let $1 \leq p, r, s < \infty$, let $w \in A_r$, and assume

$$\frac{1}{|Q|} \int_Q |f(x) - f_Q| dx \leq a(Q)$$

for every cube Q . If $a \in SD_p^s(w)$ and $p \geq rs$, then for every cube Q ,

$$\|f - f_Q\|_{\exp L(Q, \frac{w dx}{w(Q)})} \leq c_{n,p}[w]_{A_\infty} [w]_{A_r}^{\frac{1}{p}} \|a\| a(Q).$$

Idea of the proof: For every $q > p$, by using the previous case we can prove

$$\|f - f_Q\|_{L^{q,\infty}(Q, \frac{w dx}{w(Q)})} \lesssim_n q [w]_{A_\infty} [w]_{A_r}^{\frac{1}{p}} \|a\| a(Q).$$

If $\|g\|_{L^{q,\infty}} \leq \gamma q$ for every $q > p_0$, then $g \in \exp L$ and $\|g\|_{\exp L} \lesssim p_0 \gamma$.

Applications

Application 1: weighted Poincaré-Sobolev inequalities

Starting from

$$\frac{1}{|Q|} \int_Q |f - f_Q| dx \leq c_n [w]_{A_p}^{\frac{1}{p}} \ell(Q) \left(\frac{1}{w(Q)} \int_Q |\nabla f|^p w dx \right)^{\frac{1}{p}},$$

we apply the theorem with

$$a(Q) = \ell(Q) \left(\frac{1}{w(Q)} \int_Q |\nabla f|^p w dx \right)^{\frac{1}{p}} \in SD_p^n(w).$$

If $w \in A_r$, $1 \leq r \leq p < nr$, and $\frac{1}{p} - \frac{1}{p_r^*} = \frac{1}{nr}$, then

$$\begin{aligned} \|f - f_Q\|_{L^{p_r^*, \infty}(Q, \frac{w dx}{w(Q)})} &\leq c_n p_r^* [w]_{A_\infty} [w]_{A_p}^{\frac{1}{p}} [w]_{A_r}^{\frac{1}{rn}} \\ &\quad \cdot \ell(Q) \left(\frac{1}{w(Q)} \int_Q |\nabla f|^p w dx \right)^{\frac{1}{p}}. \end{aligned}$$

Application 1: weighted Poincaré-Sobolev inequalities

The truncation method upgrades the weak estimate to the Lorentz norm:

$$\|f - f_Q\|_{L^{p_r^*, p}(Q, \frac{w dx}{w(Q)})} \leq c_n p_r^*[w]_{A_\infty} [w]_{A_p}^{\frac{1}{p}} [w]_{A_r}^{\frac{1}{r_n}} \cdot \ell(Q) \left(\frac{1}{w(Q)} \int_Q |\nabla f|^p w dx \right)^{\frac{1}{p}}.$$

- The second Lorentz index p is optimal.
- In the unweighted case this is sharp even among rearrangement-invariant target spaces.
- Critical/supercritical case $p \geq nr$: exponential endpoint

$$\|f - f_Q\|_{\exp L(Q, \frac{w dx}{w(Q)})} \lesssim [w]_{A_\infty} [w]_{A_p}^{\frac{1}{p}} [w]_{A_r}^{\frac{1}{p}} \ell(Q) \left(\frac{1}{w(Q)} \int_Q |\nabla f|^p w dx \right)^{\frac{1}{p}}.$$

- The same works for pairs of weights $(u, v) \in A_p$.

Application 2: fractional Poincaré-Sobolev inequalities

We start from the weighted BBM-type Poincaré inequality of Myyryläinen–Pérez–Weigt (2024): for $w \in A_p$ and $0 < \delta < 1$,

$$\frac{1}{|Q|} \int_Q |f - f_Q| dx \leq a(Q),$$

where

$$a(Q) = c_n [w]_{A_p}^{\frac{1}{p}} (1 - \delta)^{\frac{1}{p}} \ell(Q)^\delta \left(\frac{1}{w(Q)} \int_Q \int_Q \frac{|f(x) - f(y)|^p}{|x - y|^{n + \delta p}} dy w(x) dx \right)^{\frac{1}{p}}.$$

We can prove that this functional satisfies $a \in SD_p^{n/\delta}(w)$. By the self-improving result, we obtain

$$\|f - f_Q\|_{L^{p_{\delta,r}^*}, \infty(Q, \frac{w dx}{w(Q)})} \lesssim p_{\delta,r}^*[w]_{A_\infty} [w]_{A_r}^{\frac{\delta}{nr}} a(Q),$$

where

$$\frac{1}{p} - \frac{1}{p_{\delta,r}^*} = \frac{\delta}{nr}.$$

Application 2: fractional Poincaré-Sobolev inequalities

Using the truncation method (Dyda, Lehrbäck, Vähäkangas 2023), we can improve the left-hand side to $L^{p_{\delta,r}^*,p}$ and we obtain

$$\|f - f_Q\|_{L^{p_{\delta,r}^*,p}(Q, \frac{w dx}{w(Q)})} \leq c_n p_{\delta,r}^*[w]_{A_\infty} [w]_{A_p}^{\frac{1}{p}} [w]_{A_r}^{\frac{\delta}{nr}} (1 - \delta)^{\frac{1}{p}} \\ \cdot \ell(Q)^\delta \left(\frac{1}{w(Q)} \int_Q \int_Q \frac{|f(x) - f(y)|^p}{|x - y|^{n+\delta p}} dy w(x) dx \right)^{\frac{1}{p}}.$$

- We obtain a weighted fractional Poincaré–Sobolev inequality while preserving the BBM factor $(1 - \delta)^{\frac{1}{p}}$.
- In the unweighted case, the Lorentz refinement $L^{p_{\delta}^*,p}$ is optimal among rearrangement-invariant spaces.
- The same works for pairs of weights $(u, v) \in A_p$.

Application 3: Lorentz norm of the gradient

Assume that $w \in A_r$ with $1 \leq r \leq p$. Then $w \in A_r \subset A_p = A_{p,p_r^*}$ where A_{p,p_r^*} denotes the Chung–Hunt–Kurtz class adapted to Lorentz norms.

Thus we can start with a smaller norm of the gradient

$$\begin{aligned} \frac{1}{|Q|} \int_Q |f - f_Q| \, dx &\leq c_n \ell(Q) \frac{1}{|Q|} \int_Q |\nabla f(x)| \, dx \\ &\leq c_n [w]_{A_{p,p_r^*}}^{\frac{1}{p}} \ell(Q) \|\nabla f\|_{L^{p,p_r^*}(Q, \frac{w \, dx}{w(Q)})}. \end{aligned}$$

Applying the self-improving result, we obtain

$$\begin{aligned} \|f - f_Q\|_{L^{p_r^*,\infty}(Q, \frac{w \, dx}{w(Q)})} &\leq c_n p_r^*[w]_{A_\infty} [w]_{A_{p,p_r^*}}^{\frac{1}{p}} [w]_{A_r}^{\frac{1}{rn}} \\ &\quad \cdot \ell(Q) \|\nabla f\|_{L^{p,p_r^*}(Q, \frac{w \, dx}{w(Q)})}. \end{aligned}$$

- The truncation method is not available

Further extensions

- **Higher-order oscillations:** replace f_Q by $P_Q^m f$, where $P_Q^m f$ is the projection of f onto the space of polynomials of degree at most m . This leads to Poincaré–Sobolev inequalities involving higher-order derivatives.
- **Vector-valued functions:** $f : \mathbb{R}^n \rightarrow \ell_q$, oscillation measured in the ℓ_q -norm.
- **Rectangles:** cubes replaced by rectangles with axis-parallel sides; new $SD_{p,\mathcal{R}}^s(w)$ condition adapted to rectangles.

Muchas gracias!

Local and nonlocal Laplacian diffusion on graphs with obstacles

Jaime Arto¹, Ernesto Estrada²

¹Universidad de Zaragoza

²IFISC (UIB-CSIC)

May 28, 2026

Contents

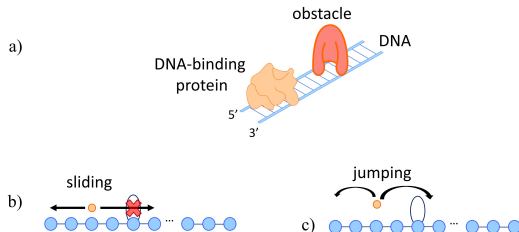
- 1 Introduction and motivation
- 2 Local diffusion: standard Laplacian
- 3 Non-local diffusion: Mellin transform
- 4 Non-local diffusion: fractional Laplacian
- 5 Another model for the obstacle

Contents

- 1 Introduction and motivation
- 2 Local diffusion: standard Laplacian
- 3 Non-local diffusion: Mellin transform
- 4 Non-local diffusion: fractional Laplacian
- 5 Another model for the obstacle

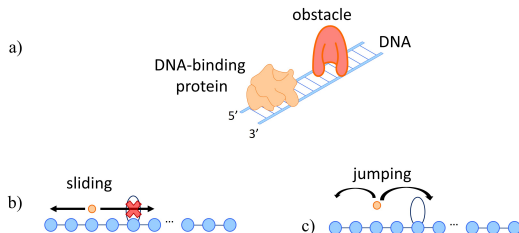
Motivation: DNA

A mass on a physical system can diffuse essentially in two ways: through-bond (slide) and through-space (jump).



Motivation: DNA

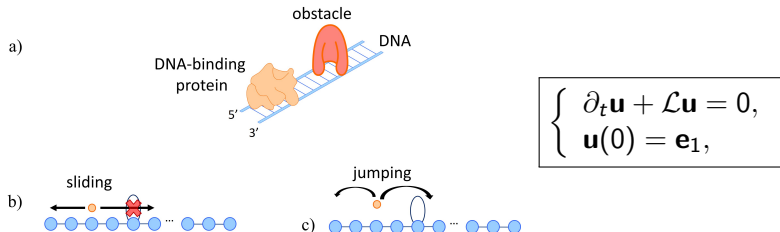
A mass on a physical system can diffuse essentially in two ways: through-bond (slide) and through-space (jump).



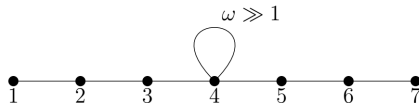
$$\begin{cases} \partial_t \mathbf{u} + \mathcal{L} \mathbf{u} = 0, \\ \mathbf{u}(0) = \mathbf{e}_1, \end{cases}$$

Motivation: DNA

A mass on a physical system can diffuse essentially in two ways: through-bond (slide) and through-space (jump).



with a diffusive operator $\mathcal{L}$ on the finite path graph P_N . The obstacle can be represented by a self-loop of large weight ω at some vertex (obtaining the graph $P_N(\omega)$).



Local and nonlocal operators

- $\mathcal{L} = L$, **standard Laplacian** If $\mathbf{u} = (u_n)_{n=1}^N$,

$$(L\mathbf{u})_n = -u_{n-1} + 2u_n - u_{n+1}, \text{ (with boundary modifications),}$$

Local and nonlocal operators

- $\mathcal{L} = L$, standard Laplacian If $\mathbf{u} = (u_n)_{n=1}^N$,

$$(\mathbf{L}\mathbf{u})_n = -u_{n-1} + 2u_n - u_{n+1}, \text{ (with boundary modifications),}$$

- $\mathcal{L} = \tilde{L}_s$, **Mellin-transformed d -path Laplacian**. If for any $d = 1, \dots, N-1$ we define the d -path Laplacian

$$(L_d \mathbf{u})_n = -u_{n-d} + 2u_n - u_{n+d} \text{ (with boundary modifications),}$$

we take

$$\tilde{L}_s \mathbf{u} = \sum_{d=1}^{N-1} \frac{1}{d^s} L_d, \quad s > 1.$$

Local and nonlocal operators

- $\mathcal{L} = L$, standard Laplacian If $\mathbf{u} = (u_n)_{n=1}^N$,

$$(\mathbf{L}\mathbf{u})_n = -u_{n-1} + 2u_n - u_{n+1}, \text{ (with boundary modifications),}$$

- $\mathcal{L} = \tilde{L}_s$, Mellin-transformed d -path Laplacian. If for any $d = 1, \dots, N-1$ we define the d -path Laplacian

$$(L_d \mathbf{u})_n = -u_{n-d} + 2u_n - u_{n+d} \text{ (with boundary modifications),}$$

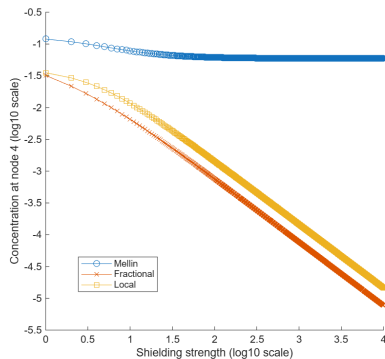
we take

$$\tilde{L}_s \mathbf{u} = \sum_{d=1}^{N-1} \frac{1}{d^s} L_d, \quad s > 1.$$

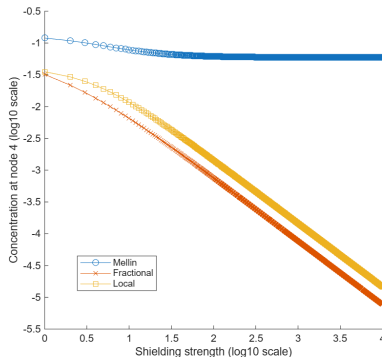
- $\mathcal{L} = L^\alpha$, **fractional Laplacian** ($\alpha \in (0, 1)$). If $L = U\Lambda U^T$ with $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_N)$, we define

$$L^\alpha = U\Lambda^\alpha U^T, \quad \Lambda^\alpha = \text{diag}(\lambda_1^\alpha, \dots, \lambda_N^\alpha)$$

Numerical results



Numerical results



The objective of the talk is to **analyze the asymptotic behavior** of each one of the operators L , L^α and $\tilde{L}_{M,s}$ on the preturbed graph $P_N(\omega)$ as $\omega \rightarrow \infty$. We generally assume that N is an odd natural number, and that the self-loop is located in the node $\frac{N+1}{2}$.

Contents

- 1 Introduction and motivation
- 2 Local diffusion: standard Laplacian
- 3 Non-local diffusion: Mellin transform
- 4 Non-local diffusion: fractional Laplacian
- 5 Another model for the obstacle

Local diffusion: standard Laplacian

$$L = D - A, \quad D = \text{degree matrix}, \quad A = \text{adjacency matrix}$$

We focus on the standard Laplacian. The usual definition on $P_N(\omega)$ is

$$L(\omega) = L + \omega \mathbf{e}_{\frac{N+1}{2}} \mathbf{e}_{\frac{N+1}{2}}^T.$$

Local diffusion: standard Laplacian

$$L = D - A, \quad D = \text{degree matrix}, \quad A = \text{adjacency matrix}$$

We focus on the standard Laplacian. The usual definition on $P_N(\omega)$ is

$$L(\omega) = L + \omega \mathbf{e}_{\frac{N+1}{2}} \mathbf{e}_{\frac{N+1}{2}}^T.$$

Why? If $\omega \in \mathbb{N}$, we are “artificially” increasing the degree of the vertex $\frac{N+1}{2}$, while the adjacency matrix remain the same.

Local diffusion: standard Laplacian

$$L = D - A, \quad D = \text{degree matrix}, \quad A = \text{adjacency matrix}$$

We focus on the standard Laplacian. The usual definition on $P_N(\omega)$ is

$$L(\omega) = L + \omega \mathbf{e}_{\frac{N+1}{2}} \mathbf{e}_{\frac{N+1}{2}}^T.$$

Why? If $\omega \in \mathbb{N}$, we are “artificially” increasing the degree of the vertex $\frac{N+1}{2}$, while the adjacency matrix remain the same.

Proposition

If $\lambda \in \sigma \left(L_{[1:\frac{N-1}{2}]} \right)$, then $\lambda \in \sigma(L) \cap \sigma(L(\omega))$

Remark

$\lambda_1 < \lambda_2 < \dots < \lambda_{\frac{N-1}{2}}$ are the *invariant eigenvalues*.

The other eigenvalues

If $\lambda \notin \sigma(L)$, the matrix determinant lemma for rank-one perturbations guarantees that

$$\begin{aligned}\det(L(\omega) - \lambda) &= \det\left(L - \lambda + \omega \mathbf{e}_{\frac{N+1}{2}} \mathbf{e}_{\frac{N+1}{2}}^T\right) \\ &= \det(L - \lambda) \left(1 + \omega ((L - \lambda)^{-1})_{\frac{N+1}{2}, \frac{N+1}{2}}\right) \\ &= \det(L - \lambda) \left(1 + \omega \frac{\det\left(L_{[1:\frac{N+1}{2}]} - \lambda\right)}{\det(L - \lambda)}\right)\end{aligned}$$

The other eigenvalues

If $\lambda \notin \sigma(L)$, the matrix determinant lemma for rank-one perturbations guarantees that

$$\begin{aligned}\det(L(\omega) - \lambda) &= \det\left(L - \lambda + \omega \mathbf{e}_{\frac{N+1}{2}} \mathbf{e}_{\frac{N+1}{2}}^T\right) \\ &= \det(L - \lambda) \left(1 + \omega ((L - \lambda)^{-1})_{\frac{N+1}{2}, \frac{N+1}{2}}\right) \\ &= \det(L - \lambda) \left(1 + \omega \frac{\det\left(L_{[1:\frac{N+1}{2}]} - \lambda\right)}{\det(L - \lambda)}\right)\end{aligned}$$

Therefore, $\lambda \in \sigma(L(\omega))$ if and only if

$$g(\lambda) := \frac{\det(L - \lambda)}{\det\left(L_{[1:\frac{N+1}{2}]} - \lambda\right)} = -\omega.$$

The other eigenvalues

Since $g(\lambda)$ has simple poles at every λ_i , $i = 1, \dots, \frac{N-1}{2}$, by Bolzano's theorem there exist $\lambda_i^{(\omega)}$, $i = 1, \dots, \frac{N-3}{2}$ such that $g\left(\lambda_i^{(\omega)}\right) = 0$ with $\lambda_i < \lambda_{i+1}^{(\omega)} < \lambda_{i+1}$.

The other eigenvalues

Since $g(\lambda)$ has simple poles at every λ_i , $i = 1, \dots, \frac{N-1}{2}$, by Bolzano's theorem there exist $\lambda_i^{(\omega)}$, $i = 1, \dots, \frac{N-3}{2}$ such that $g\left(\lambda_i^{(\omega)}\right) = 0$ with $\lambda_i < \lambda_{i+1}^{(\omega)} < \lambda_{i+1}$.

Moreover, it is easy to see that there exist other eigenvalues $\lambda_1^{(\omega)} \in (0, \lambda_1)$ and $\lambda^{(\omega)} \in [\omega, \omega + 4]$ (by Gerschgorin's disk theorem). We may then order the eigenvalues of $L(\omega)$ as

$$\lambda_1^{(\omega)} < \lambda_1 < \lambda_2^{(\omega)} < \lambda_2 < \dots < \lambda_{\frac{N-1}{2}}^{(\omega)} < \lambda_{\frac{N-1}{2}} < \lambda^{(\omega)}.$$

The other eigenvalues

Since $g(\lambda)$ has simple poles at every λ_i , $i = 1, \dots, \frac{N-1}{2}$, by Bolzano's theorem there exist $\lambda_i^{(\omega)}$, $i = 1, \dots, \frac{N-3}{2}$ such that $g\left(\lambda_i^{(\omega)}\right) = 0$ with $\lambda_i < \lambda_{i+1}^{(\omega)} < \lambda_{i+1}$.

Moreover, it is easy to see that there exist other eigenvalues $\lambda_1^{(\omega)} \in (0, \lambda_1)$ and $\lambda^{(\omega)} \in [\omega, \omega + 4]$ (by Gerschgorin's disk theorem). We may then order the eigenvalues of $L(\omega)$ as

$$\lambda_1^{(\omega)} < \lambda_1 < \lambda_2^{(\omega)} < \lambda_2 < \dots < \lambda_{\frac{N-1}{2}}^{(\omega)} < \lambda_{\frac{N-1}{2}} < \lambda^{(\omega)}.$$

Proposition

For every $j = 1, \dots, \frac{N-1}{2}$ one has that

$$\lambda_j^{(\omega)} \rightarrow \lambda_j, \quad \omega \rightarrow \infty,$$

and $\lambda^{(\omega)} \rightarrow \infty$.

Behavior of eigenvectors

A bisymmetric matrix of odd dimension N has $\frac{N+1}{2}$ symmetric eigenvectors and $\frac{N-1}{2}$ antisymmetric eigenvectors

- If $L_{[1:\frac{N-1}{2}]} \mathbf{v}_j = \lambda_j \mathbf{v}_j$, then the (normalized) eigenvector of $L(\omega)$ associated with λ_j is

$$\mathbf{w}_j = \begin{pmatrix} \mathbf{v}_j \\ 0 \\ -R\mathbf{v}_j \end{pmatrix}$$

Behavior of eigenvectors

A bisymmetric matrix of odd dimension N has $\frac{N+1}{2}$ symmetric eigenvectors and $\frac{N-1}{2}$ antisymmetric eigenvectors

- If $L_{[1:\frac{N-1}{2}]} \mathbf{v}_j = \lambda_j \mathbf{v}_j$, then the (normalized) eigenvector of $L(\omega)$ associated with λ_j is

$$\mathbf{w}_j = \begin{pmatrix} \mathbf{v}_j \\ 0 \\ -R\mathbf{v}_j \end{pmatrix}$$

- Therefore, the (normalized) eigenvectors of $L(\omega)$ associated with $\lambda_j^{(\omega)}$ and $\lambda^{(\omega)}$ are

$$\mathbf{w}_j^{(\omega)} = \begin{pmatrix} \mathbf{u}_j^{(\omega)} \\ a_j^{(\omega)} \\ R\mathbf{u}_j^{(\omega)} \end{pmatrix}, \quad \mathbf{w}^{(\omega)} = \begin{pmatrix} \mathbf{u}^{(\omega)} \\ a^{(\omega)} \\ R\mathbf{u}^{(\omega)} \end{pmatrix}$$

Behavior of eigenvectors

Proposition

We have that $\lim_{\omega \rightarrow \infty} a_j^{(\omega)} = 0$ and the componentwise limit

$$\lim_{\omega \rightarrow \infty} \mathbf{u}_j^{(\omega)} = \mathbf{v}_j.$$

Sketch of the proof.

It can be easily seen that

$$a_j^{(\omega)} = \frac{-2 \left(\mathbf{u}_j^{(\omega)} \right)^{\frac{N-1}{2}}}{\lambda_j^{(\omega)} - 2 - \omega} \rightarrow 0, \quad \omega \rightarrow \infty.$$

Behavior of eigenvectors

Proposition

We have that $\lim_{\omega \rightarrow \infty} a_j^{(\omega)} = 0$ and the componentwise limit

$$\lim_{\omega \rightarrow \infty} \mathbf{u}_j^{(\omega)} = \mathbf{v}_j.$$

Sketch of the proof.

It can be easily seen that

$$a_j^{(\omega)} = \frac{-2 \left(\mathbf{u}_j^{(\omega)} \right)_{\frac{N-1}{2}}}{\lambda_j^{(\omega)} - 2 - \omega} \rightarrow 0, \quad \omega \rightarrow \infty.$$

Furthermore, one can check that

$$\left(L_{[1:\frac{N-1}{2}]} - \lambda_j^{(\omega)} \right) \mathbf{u}_j^{(\omega)} = a_j^{(\omega)} \hat{\mathbf{e}}_{\frac{N-1}{2}} \rightarrow \mathbf{0}_{\frac{N-1}{2}}, \quad \omega \rightarrow \infty,$$

and hence $\lim_{\omega \rightarrow \infty} \mathbf{u}_j^{(\omega)} = \mathbf{v}_j$



Asymptotic behavior of the solution

Theorem

Take $i \in (\frac{N+1}{2}, N]$ and $t > 0$. If $\mathcal{L} = L(\omega)$, then

$$\lim_{\omega \rightarrow \infty} \left(\mathbf{u}^{(\omega)}(t) \right)_i = 0.$$

Proof.

$$\begin{aligned} \left(\mathbf{u}^{(\omega)}(t) \right)_i &= \left(e^{-tL_1(\omega)} \mathbf{e}_1 \right)_i \\ &= \sum_{j=1}^{\frac{N-1}{2}} \left(\left(\mathbf{w}_j \mathbf{w}_j^T \mathbf{e}_1 \right)_i e^{-t\lambda_j} + \left(\mathbf{w}_j^{(\omega)} \left(\mathbf{w}_j^{(\omega)} \right)^T \mathbf{e}_1 \right)_i e^{-t\lambda_j^{(\omega)}} + \left(\mathbf{w}^{(\omega)} \left(\mathbf{w}^{(\omega)} \right)^T \mathbf{e}_1 \right)_i e^{-t\lambda^{(\omega)}} \right) \\ &= \sum_{j=1}^{\frac{N-1}{2}} \left(-(\mathbf{v}_j)_{N-i+1} (\mathbf{v}_j)_1 e^{-t\lambda_j} + \left(\mathbf{u}_j^{(\omega)} \right)_{N-i+1} \left(\mathbf{u}_j^{(\omega)} \right)_1 e^{-t\lambda_j^{(\omega)}} \right) \\ &\quad + \left(\mathbf{w}^{(\omega)} \left(\mathbf{w}^{(\omega)} \right)^T \right)_{i1} e^{-t\lambda^{(\omega)}}. \end{aligned}$$

Asymptotic behavior of the solution

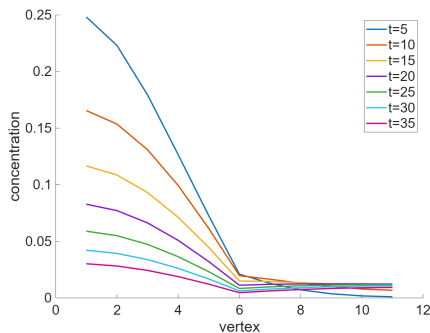
Proof.

Since $\mathbf{w}^{(\omega)}$ is bounded (normalized), $\lambda_j^{(\omega)} \rightarrow \lambda_j$ and $\lambda^{(\omega)} \rightarrow \infty$, we have that

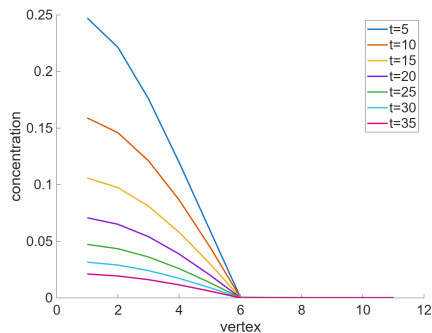
$$\begin{aligned} \left(\mathbf{u}^{(\omega)}(t) \right)_i &= \sum_{j=1}^{\frac{N-1}{2}} \left(-(\mathbf{v}_j)_{N-i+1} (\mathbf{v}_j)_1 e^{-t\lambda_j} + \left(\mathbf{u}_j^{(\omega)} \right)_{N-i+1} \left(\mathbf{u}_j^{(\omega)} \right)_1 e^{-t\lambda_j^{(\omega)}} \right) \\ &\quad + \left(\mathbf{w}^{(\omega)} \left(\mathbf{w}^{(\omega)} \right)^T \right)_{i1} e^{-t\lambda^{(\omega)}} \\ &\xrightarrow{\omega \rightarrow \infty} \sum_{j=1}^{\frac{N-1}{2}} \left(-(\mathbf{v}_j)_{N-i+1} (\mathbf{v}_j)_1 e^{-t\lambda_j} + (\mathbf{v}_j)_{N-i+1} (\mathbf{v}_j)_1 e^{-t\lambda_j} \right) = 0 \end{aligned}$$



Numerical experiments



(a) $\omega = 2$



(b) $\omega = 100$

Contents

- 1 Introduction and motivation
- 2 Local diffusion: standard Laplacian
- 3 Non-local diffusion: Mellin transform
- 4 Non-local diffusion: fractional Laplacian
- 5 Another model for the obstacle

Non-local diffusion: Mellin transform

For $d > 1$ one has $L_d(\omega) = L_d$ ($L_d = D_d - A_d$), so

$$\tilde{L}_{M,s}(\omega) = \sum_{d=1}^{N-1} \frac{1}{d^s} L_d(\omega) = \sum_{d=1}^{N-1} \frac{1}{d^s} L_d + \omega \mathbf{e}_1 \mathbf{e}_1^T = \tilde{L}_{M,s} + \omega \mathbf{e}_1 \mathbf{e}_1^T$$

Non-local diffusion: Mellin transform

For $d > 1$ one has $L_d(\omega) = L_d (L_d = D_d - A_d)$, so

$$\tilde{L}_{M,s}(\omega) = \sum_{d=1}^{N-1} \frac{1}{d^s} L_d(\omega) = \sum_{d=1}^{N-1} \frac{1}{d^s} L_d + \omega \mathbf{e}_1 \mathbf{e}_1^T = \tilde{L}_{M,s} + \omega \mathbf{e}_1 \mathbf{e}_1^T$$

Proposition

Write

$$\tilde{L}_{M,s} = \left(\begin{array}{c|c|c} \tilde{B} & \mathbf{x} & \tilde{C} \\ \hline \mathbf{x}^T & x & \mathbf{x}^T R \\ \hline \tilde{C}^T & R\mathbf{x} & R\tilde{B}R \end{array} \right).$$

Then there exists an orthogonal matrix P such that

$$P^T \tilde{L}_{M,s} P = \left(\begin{array}{c|c} L^+ & \\ \hline & L^- \end{array} \right), \quad P^T \tilde{L}_{M,s}(\omega) P = \left(\begin{array}{c|c} L_\omega^+ & \\ \hline & L^- \end{array} \right)$$

where

$$L^- = R\tilde{B}R - R\tilde{C}, \quad L^+ = \left(\begin{array}{c|c} L^{++} & \mathbf{y} \\ \hline \mathbf{y}^T & x \end{array} \right), \quad L_\omega^+ = \left(\begin{array}{c|c} L^{++} & \mathbf{y} \\ \hline \mathbf{y}^T & x + \omega \end{array} \right),$$

with

$$L^{++} = \tilde{B} + \tilde{C}R, \quad \mathbf{y} = \sqrt{2}\mathbf{x}.$$

Expression of the limit

Using similar arguments as before one can show that

$$\lim_{\omega \rightarrow \infty} \left(\mathbf{u}^{(\omega)}(t) \right)_i = \frac{1}{2} \mathbf{e}_i^T \left(\begin{array}{c|c|c} e^{-tL^{++}} + Re^{-tL^-} R & \mathbf{0}_{\frac{N-1}{2}} & e^{-tL^{++}} R - Re^{-tL^-} \\ \hline \mathbf{0}_{\frac{N-1}{2}}^T & 0 & \mathbf{0}_{\frac{N-1}{2}}^T \\ \hline Re^{-tL^{++}} - e^{-tL^-} R & \mathbf{0}_{\frac{N-1}{2}} & Re^{-tL^{++}} R + e^{-tL^-} \end{array} \right) \mathbf{e}_1.$$

Expression of the limit

Using similar arguments as before one can show that

$$\lim_{\omega \rightarrow \infty} \left(\mathbf{u}^{(\omega)}(t) \right)_i = \frac{1}{2} \mathbf{e}_i^T \left(\begin{array}{c|c|c} e^{-tL^{++}} + Re^{-tL^-} R & \mathbf{0}_{\frac{N-1}{2}} & e^{-tL^{++}} R - Re^{-tL^-} \\ \hline \mathbf{0}_{\frac{N-1}{2}}^T & 0 & \mathbf{0}_{\frac{N-1}{2}}^T \\ \hline Re^{-tL^{++}} - e^{-tL^-} R & \mathbf{0}_{\frac{N-1}{2}} & Re^{-tL^{++}} R + e^{-tL^-} \end{array} \right) \mathbf{e}_1.$$

Therefore, we are interested in the first columns of the matrices

$$\frac{1}{2} \left(e^{-tL^{++}} + Re^{-tL^-} R \right) \quad (\text{before the barrier}),$$

and

$$\frac{1}{2} \left(Re^{-tL^{++}} - e^{-tL^-} R \right) \quad (\text{after the barrier}).$$

Expression of the limit

Using similar arguments as before one can show that

$$\lim_{\omega \rightarrow \infty} \left(\mathbf{u}^{(\omega)}(t) \right)_i = \frac{1}{2} \mathbf{e}_i^T \left(\begin{array}{c|c|c} e^{-tL^{++}} + Re^{-tL^-} R & \mathbf{0}_{\frac{N-1}{2}} & e^{-tL^{++}} R - Re^{-tL^-} \\ \hline \mathbf{0}_{\frac{N-1}{2}}^T & 0 & \mathbf{0}_{\frac{N-1}{2}}^T \\ \hline Re^{-tL^{++}} - e^{-tL^-} R & \mathbf{0}_{\frac{N-1}{2}} & Re^{-tL^{++}} R + e^{-tL^-} \end{array} \right) \mathbf{e}_1.$$

Therefore, we are interested in the first columns of the matrices

$$\frac{1}{2} \left(e^{-tL^{++}} + Re^{-tL^-} R \right) \quad (\text{before the barrier}),$$

and

$$\frac{1}{2} \left(Re^{-tL^{++}} - e^{-tL^-} R \right) \quad (\text{after the barrier}).$$

We are particularly interested in

$$\frac{1}{2} \left(Re^{-tL^{++}} - e^{-tL^-} R \right) = \frac{1}{2} \left(e^{-tRL^{++}R} - e^{-tL^-} \right) R$$

Key lemmas

Objective: prove $e^{-tRL^{++}R} - e^{-tL^-} > 0$.

Key lemmas

Objective: prove $e^{-tRL^{++}R} - e^{-tL^-} > 0$.

Lemma (Duhamel's formula)

$$e^M - e^Q = \int_0^1 e^{(1-x)Q} (M - Q) e^{xM} dx, \quad M, Q > 0.$$

Key lemmas

Objective: prove $e^{-tRL^{++}R} - e^{-tL^-} > 0$.

Lemma (Duhamel's formula)

$$e^M - e^Q = \int_0^1 e^{(1-x)Q} (M - Q) e^{xM} dx, \quad M, Q > 0.$$

Lemma

$$-t(RL^{++}R - L^-) > 0, \quad t > 0.$$

Proof.

$$RL^{++}R - L^- = 2R\tilde{C} < 0.$$



Proposition

The matrices $-tRL^{++}R$ and $-tL^-$ are strictly Metzler.

Proof.

$$RL^{++}R = R \left(\tilde{B} + R\tilde{C} \right) R.$$
$$\tilde{B}_{i,j} < 0, \quad i \neq j$$

Proposition

The matrices $-tRL^{++}R$ and $-tL^-$ are strictly Metzler.

Proof.

$$RL^{++}R = R \left(\tilde{B} + R\tilde{C} \right) R.$$

$$\tilde{B}_{i,j} < 0, \quad i \neq j$$

$$\tilde{C}_{i,j} < 0, \quad \forall i,j$$

Proposition

The matrices $-tRL^{++}R$ and $-tL^-$ are strictly Metzler.

Proof.

$$RL^{++}R = R \left(\tilde{B} + R\tilde{C} \right) R.$$
$$\left. \begin{array}{l} \tilde{B}_{i,j} < 0, \quad i \neq j \\ \tilde{C}_{i,j} < 0, \quad \forall i,j \end{array} \right\} \Rightarrow (\tilde{B} + R\tilde{C})_{i,j} < 0, \quad i \neq j.$$

Proposition

The matrices $-tRL^{++}R$ and $-tL^-$ are strictly Metzler.

Proof.

$$RL^{++}R = R(\tilde{B} + R\tilde{C})R.$$

$$\left. \begin{array}{l} \tilde{B}_{i,j} < 0, \quad i \neq j \\ \tilde{C}_{i,j} < 0, \quad \forall i, j \end{array} \right\} \Rightarrow (\tilde{B} + R\tilde{C})_{i,j} < 0, \quad i \neq j.$$

$$(L^-)_{i,j} = (R\tilde{B}R - R\tilde{C})_{i,j} = \frac{-1}{(j-i)^s} + \frac{1}{(j+i)^s} < 0, \quad i \neq j. \quad \square$$

Asymptotic behavior of the solution

$$-t(RL^{++}R - L^-) > 0, \quad -tRL^{++}R \text{ and } -tL^- \text{ strictly Metzler}$$

Theorem

Take $i \in (\frac{N+1}{2}, N]$ and $t > 0$. If $\mathcal{L} = \tilde{L}_{M,s}(\omega)$, then

$$\lim_{\omega \rightarrow \infty} \left(\mathbf{u}^{(\omega)}(t) \right)_i > 0.$$

Proof.

By Duhamel's formula

$$e^{-tRL^{++}R} - e^{-tL^-} = \int_0^1 e^{(1-x)(-tL^-)} (-t(RL^{++}R - L^-)) e^{x(-tRL^{++}R)} dx.$$

Asymptotic behavior of the solution

$$-t(RL^{++}R - L^-) > 0, \quad -tRL^{++}R \text{ and } -tL^- \text{ strictly Metzler}$$

Theorem

Take $i \in (\frac{N+1}{2}, N]$ and $t > 0$. If $\mathcal{L} = \tilde{L}_{M,s}(\omega)$, then

$$\lim_{\omega \rightarrow \infty} \left(\mathbf{u}^{(\omega)}(t) \right)_i > 0.$$

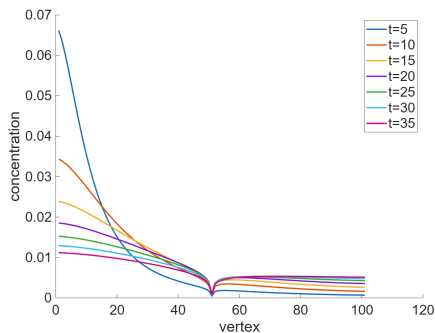
Proof.

By Duhamel's formula

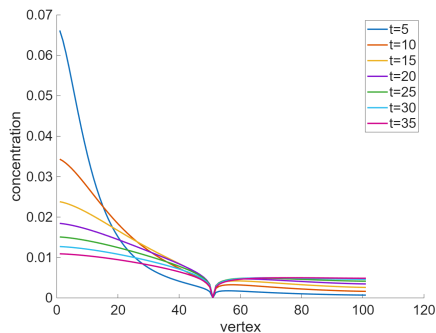
$$e^{-tRL^{++}R} - e^{-tL^-} = \int_0^1 e^{(1-x)(-tL^-)} (-t(RL^{++}R - L^-)) e^{x(-tRL^{++}R)} dx.$$

$e^{x(-tRL^{++}R)} > 0$, $e^{(1-x)(-tL^-)} > 0$, so the integral is strictly positive. $\square$

Numerical experiments



(a) $\omega = 10$



(b) $\omega = 100$

Contents

- 1 Introduction and motivation
- 2 Local diffusion: standard Laplacian
- 3 Non-local diffusion: Mellin transform
- 4 Non-local diffusion: fractional Laplacian**
- 5 Another model for the obstacle

Non-local diffusion: fractional Laplacian

Finally, we focus on the fractional Laplacian.

$$L^\alpha(\omega) := (L(\omega))^\alpha = \left(L + \omega \mathbf{e}_1 \mathbf{e}_1^T\right)^\alpha$$

Remark

$$\left(L + \omega \mathbf{e}_1 \mathbf{e}_1^T\right)^\alpha \neq L^\alpha + \omega \mathbf{e}_1 \mathbf{e}_1^T$$

Theorem

Take $i \in \left(\frac{N+1}{2}, N\right]$, $\alpha \in (0, 1)$ and $t > 0$. If $\mathcal{L} = L^\alpha(\omega)$, then

$$\lim_{\omega \rightarrow \infty} \left(\mathbf{u}^{(\omega)}(t)\right)_i = 0.$$

Non-local diffusion: fractional Laplacian

Eigenvalues of $L(\omega)$: $\lambda_1^{(\omega)} < \lambda_1 < \lambda_2^{(\omega)} < \lambda_2 < \dots < \lambda_{\frac{N-1}{2}}^{(\omega)} < \lambda_{\frac{N-1}{2}} < \lambda^{(\omega)}$

Proof.

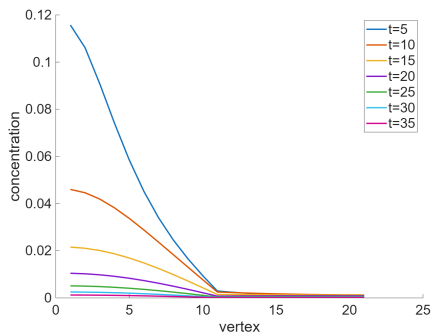
The eigenvalues of $L^\alpha(\omega)$ are

$$\left(\lambda_1^{(\omega)}\right)^\alpha < \lambda_1^\alpha < \left(\lambda_2^{(\omega)}\right)^\alpha < \lambda_2^\alpha < \dots < \left(\lambda_{\frac{N-1}{2}}^{(\omega)}\right)^\alpha < \lambda_{\frac{N-1}{2}}^\alpha < \left(\lambda^{(\omega)}\right)^\alpha,$$

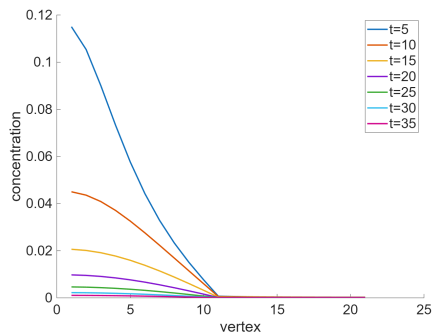
and the eigenvectors are those of $L(\omega)$. Therefore

$$\begin{aligned} \left(\mathbf{u}^{(\omega)}(t)\right)_i &= \sum_{j=1}^{\frac{N-1}{2}} \left(-(\mathbf{v}_j)_{N-i+1} (\mathbf{v}_j)_1 e^{-t\lambda_j^\alpha} + \left(\mathbf{u}_j^{(\omega)}\right)_{N-i+1} \left(\mathbf{u}_j^{(\omega)}\right)_1 e^{-t\left(\lambda_j^{(\omega)}\right)^\alpha} \right) \\ &+ \left(\mathbf{w}^{(\omega)} \left(\mathbf{w}^{(\omega)} \right)^T \right)_{i1} e^{-t\left(\lambda^{(\omega)}\right)^\alpha} \\ &\xrightarrow{\omega \rightarrow \infty} \sum_{j=1}^{\frac{N-1}{2}} \left(-(\mathbf{v}_j)_{N-i+1} (\mathbf{v}_j)_1 e^{-t\lambda_j^\alpha} + (\mathbf{v}_j)_{N-i+1} (\mathbf{v}_j)_1 e^{-t\lambda_j^\alpha} \right) = 0 \quad \square \end{aligned}$$

Numerical experiments



(a) $\omega = 2$



(b) $\omega = 10$

Generalization

Given $f : [0, \infty) \longrightarrow \mathbb{R}$ and $L = U\Lambda U^T$ with $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_N)$,

$$f(L) = Uf(\Lambda)U^T, \quad f(\Lambda) = \text{diag}(f(\lambda_1), \dots, f(\lambda_N)).$$

Given $f : [0, \infty) \rightarrow \mathbb{R}$ and $L = U\Lambda U^T$ with $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_N)$,

$$f(L) = Uf(\Lambda)U^T, \quad f(\Lambda) = \text{diag}(f(\lambda_1), \dots, f(\lambda_N)).$$

Theorem

Take $i \in (\frac{N+1}{2}]$ and $t > 0$. If $\mathcal{L} = f(L)$ for some continuous and bounded-below function, then

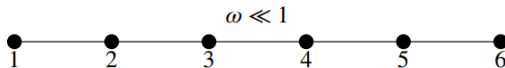
$$\lim_{\omega \rightarrow \infty} \left(\mathbf{u}^{(\omega)}(t) \right)_i = 0.$$

Contents

- 1 Introduction and motivation
- 2 Local diffusion: standard Laplacian
- 3 Non-local diffusion: Mellin transform
- 4 Non-local diffusion: fractional Laplacian
- 5 Another model for the obstacle

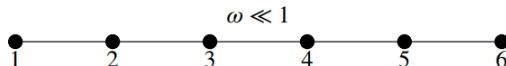
Another model

Given a path graph of N vertices, we can model an obstacle by weakening the weight of one of the vertices.

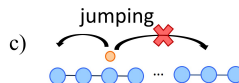
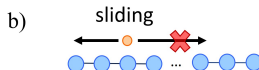
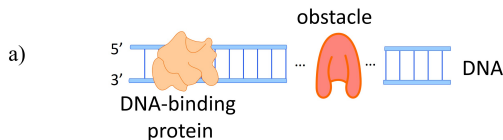


Another model

Given a path graph of N vertices, we can model an obstacle by wakening the weight of one of the vertices.



In the DNA example, a very small weight may hinder both through-bond and through-space diffusion (we are splitting the graph in two).

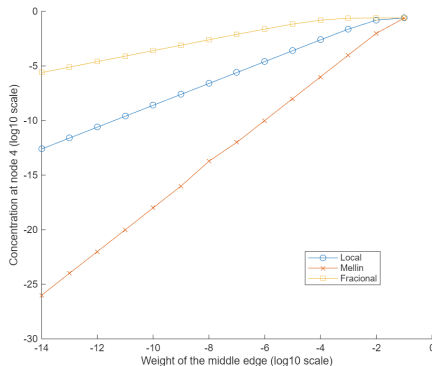


Reason

Now the distance of the shortest path P is defined as

$$\ell(P) = \sum_{i \in P} \frac{1}{w(v_{i-1}, v_i)}.$$

This will dramatically increase the distance between two vertices at different sides of the barrier (future work: prove this analytically)



Thx for your attention :)

Singular Inner Multipliers and Cyclic Vectors for Coefficient and Besov Spaces

Alberto Dayan
(joint with D. Seco)

Universitat Autònoma de Barcelona

EARCO 2026
May 28, 2026

The Commutant of the Shift

Let $S: \ell^2 \rightarrow \ell^2$ be the **shift operator**

$$S: (a_0, a_1, a_2, \dots) \mapsto (0, a_0, a_1, \dots)$$

Problem (Wiener, 1920): find all bounded, linear maps $T: \ell^2 \rightarrow \ell^2$ such that $TS = ST$.

The Commutant of the Shift

Let $S: \ell^2 \rightarrow \ell^2$ be the **shift operator**

$$S: (a_0, a_1, a_2, \dots) \mapsto (0, a_0, a_1, \dots)$$

Problem (Wiener, 1920): find all bounded, linear maps $T: \ell^2 \rightarrow \ell^2$ such that $TS = ST$.

Let H^2 be the **Hardy space**

$$H^2 := \left\{ f \in \text{Hol}(\mathbb{D}) \mid \sum_{n \geq 0} |\hat{f}(n)|^2 < \infty \right\}$$

Then S is unitarily equivalent to

$$M_z: f \mapsto z f \quad f \in H^2$$

Wiener's Theorem

Problem: find all $T \in \mathcal{B}(H^2)$ such that $TM_z = M_z T$

Example

Let ϕ be in

$$H^\infty := \left\{ f \in \text{Hol}(\mathbb{D}) \mid \sup_{z \in \mathbb{D}} |f(z)| < \infty \right\}.$$

Then $TM_z = M_z T$ and

$$M_\phi: f \mapsto \phi f$$

is bounded, since $H^2 = L^2(\mathbb{T}, dm) \cap \text{Hol}(\mathbb{D})$.

Wiener's Theorem

Problem: find all $T \in \mathcal{B}(H^2)$ such that $TM_z = M_z T$

Example

Let ϕ be in

$$H^\infty := \left\{ f \in \text{Hol}(\mathbb{D}) \mid \sup_{z \in \mathbb{D}} |f(z)| < \infty \right\}.$$

Then $TM_z = M_z T$ and

$$M_\phi: f \mapsto \phi f$$

is bounded, since $H^2 = L^2(\mathbb{T}, dm) \cap \text{Hol}(\mathbb{D})$.

Theorem (Wiener, 1930)

A linear map $T: H^2 \rightarrow H^2$ is bounded and commutes with M_z if and only if $T = M_\phi$ for some ϕ in H^∞ .

Coefficient Spaces

For all $p \geq 1$, let

$$\ell_A^p := \left\{ f \in \text{Hol}(\mathbb{D}) \mid \sum_{n \geq 0} |\hat{f}(n)|^p < \infty \right\}$$

Definition

$$\mathcal{M}_p := \{ \phi \in \text{Hol}(\mathbb{D}) \mid \phi f \in \ell_A^p, \text{ for all } f \in \ell_A^p \}$$

Theorem (Nikolski, 1966)

$T: \ell_A^p \rightarrow \ell_A^p$ is bounded, linear and $TM_z = M_z T$ if and only if $T = M_\phi$, for some $\phi \in \mathcal{M}_p$.

Properties of Multipliers

Open problem: describe $\mathcal{M}_p$, $p \notin \{1, 2\}$.

- $\mathcal{M}_2 = H^\infty$.
- $\mathcal{M}_1 = \ell_A^1$ (**Wiener algebra**)
- $\ell_A^1 \subset \mathcal{M}_p \subset \ell_A^p \cap H^\infty$.

Properties of Multipliers

Open problem: describe $\mathcal{M}_p$, $p \notin \{1, 2\}$.

- $\mathcal{M}_2 = H^\infty$.
- $\mathcal{M}_1 = \ell_A^1$ (**Wiener algebra**)
- $\ell_A^1 \subset \mathcal{M}_p \subset \ell_A^p \cap H^\infty$.

Theorem (Nikolski, 1966)

$$\mathcal{M}_p = \mathcal{M}_q, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Question

Let $p \geq 1$. Which **inner functions** Θ are in $\mathcal{M}_p$?

Inner functions

A function Θ is **inner** if:

$$\lim_{r \rightarrow 1} |\Theta(r\xi)| = 1 \quad a.e. \xi \in \mathbb{T}.$$

- Any inner function is an **isometric multiplier of H^2** .
- Any inner function Θ can be factorized as

$$\Theta = S_\mu B,$$

where S_μ is a (non vanishing) **singular inner function**

$$S_\mu(z) := \exp \left\{ - \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} d\mu(\xi) \right\} \quad z \in \mathbb{D},$$

and B is a **Blaschke product**.

Beurling's theorem

Theorem (Beurling, '49)

Any closed, $\{0\} \neq M \subsetneq H^2$ which is **shift-invariant**

$$Mz(M) \subset M$$

is of the form

$$M = \Theta H^2$$

for some inner function $\Theta \neq 1$.

Beurling's theorem

Theorem (Beurling, '49)

Any closed, $\{0\} \neq M \subsetneq H^2$ which is **shift-invariant**

$$Mz(M) \subset M$$

is of the form

$$M = \Theta H^2$$

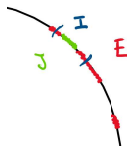
for some inner function $\Theta \neq 1$.

- (Vinogradov, Lebedev, Olewski, ...) Blaschke products in $\mathcal{M}_p$ depending on its zeros.
- To this day, there isn't a single example of a singular inner function $S_\mu \in \mathcal{M}_p$, if $p \neq 2$!

Concentrated Measures

$E \subset \mathbb{T}$, $I \subset \mathbb{T}$ interval

$$d_E(I) := \sup\{|J| \mid J \subseteq I \text{ interval}, J \cap E = \emptyset\}$$



Theorem (Lebedev, 1998)

If $\mu(E) > 0$ for some $E \subset \mathbb{T}$ satisfying

$$\inf_I \frac{d_E(I)}{|I|} = \delta > 0,$$

then $S_\mu \notin \mathcal{M}_p$ for all $p \neq 2$

Example

$$\mu = \delta_{\{1\}} \implies S_\mu(z) = e^{\frac{1+z}{z-1}} \notin \mathcal{M}_p \text{ for all } p \neq 2.$$

Cyclicity

For $f \in \ell_A^p$,

$$[f]_p := \overline{\{pf \mid p \text{ polynomial}\}}^{\ell_A^p}$$

is the smallest closed M_z -invariant $M \subset \ell_A^p$ that contains f .

Definition

f is **cyclic** in ℓ_A^p if $[f]_p = \ell_A^p$

Cyclicity

For $f \in \ell_A^p$,

$$[f]_p := \overline{\{pf \mid p \text{ polynomial}\}}^{\ell_A^p}$$

is the smallest closed M_z -invariant $M \subset \ell_A^p$ that contains f .

Definition

f is **cyclic** in ℓ_A^p if $[f]_p = \ell_A^p$

- f cyclic $\implies f$ non vanishing.
- No inner function is cyclic in ℓ_A^p , for all $p \leq 2$.

Question

Can a singular inner function S_μ be cyclic in ℓ_A^p , for $p > 2$?

A Necessary Condition for $S_\mu \in \mathcal{M}_p$

Lemma

Let $p > 2$ and S_μ be a singular inner function and assume that S_μ is cyclic in ℓ_A^p . Then $S_\mu \notin \mathcal{M}_p$.

Proof.

Suppose that $S_\mu \in \mathcal{M}_p$. Then $S_\mu \in \mathcal{M}_q \subset \ell_A^q$.

A Necessary Condition for $S_\mu \in \mathcal{M}_p$

Lemma

Let $p > 2$ and S_μ be a singular inner function and assume that S_μ is cyclic in ℓ_A^p . Then $S_\mu \notin \mathcal{M}_p$.

Proof.

Suppose that $S_\mu \in \mathcal{M}_p$. Then $S_\mu \in \mathcal{M}_q \subset \ell_A^q$. Therefore,

$$\Phi_\mu(f) := \int_{\mathbb{T}} \xi f(\xi) \overline{S_\mu(\xi)} d\xi$$

is a non-zero functional in $(\ell_A^p)^*$.

A Necessary Condition for $S_\mu \in \mathcal{M}_p$

Lemma

Let $p > 2$ and S_μ be a singular inner function and assume that S_μ is cyclic in ℓ_A^p . Then $S_\mu \notin \mathcal{M}_p$.

Proof.

Suppose that $S_\mu \in \mathcal{M}_p$. Then $S_\mu \in \mathcal{M}_q \subset \ell_A^q$. Therefore,

$$\Phi_\mu(f) := \int_{\mathbb{T}} \xi f(\xi) \overline{S_\mu(\xi)} d\xi$$

is a non-zero functional in $(\ell_A^p)^*$. Since

$$\Phi_\mu(z^m S_\mu) = \int_{\mathbb{T}} z^{m+1} d\xi = 0 \quad m = 0, 1, 2, \dots,$$

$\Phi_\mu([S_\mu]_p) = \{0\}$, hence S_μ is not cyclic in ℓ_A^p .

Smooth Measures

If μ is a positive measure on $\mathbb{T}$, set

$$\omega_{\mu}(t) := \sup_{\substack{l_i \text{ adjacent} \\ |l_1|=|l_2|\leq t}} |\mu(l_1) - \mu(l_2)|.$$

For instance, $\omega_{\delta_{\{1\}}} = 1$ and $\omega_{dm} = 0$.

Smooth Measures

If μ is a positive measure on $\mathbb{T}$, set

$$\omega_{\mu}(t) := \sup_{\substack{I_i \text{ adjacent} \\ |I_1|=|I_2|\leq t}} |\mu(I_1) - \mu(I_2)|.$$

For instance, $\omega_{\delta_{\{1\}}} = 1$ and $\omega_{dm} = 0$.

Properties

- (i) **(Stein & Zygmund, 1964)** If $\int_0^1 \frac{\omega_{\mu}(t)^2}{t^3} dt < \infty$, then μ is absolutely continuous.
- (ii) **(Shapiro, 1968)** There exists a **singular** measure μ such that

$$\omega_{\mu}(t) \leq C \frac{t}{\sqrt{\log \frac{e}{t}}}$$

Cyclic Singular Inner Vectors

Theorem (DS, '26)

Let μ be a singular measure on $\mathbb{T}$. If

$$\omega_\mu(t) \leq C \frac{t}{\sqrt{\log \frac{e}{t}}},$$

then S_μ is cyclic in ℓ_A^p for all $p > 2$. In particular, $S_\mu \notin \mathcal{M}_p$.

Cyclic Singular Inner Vectors

Theorem (DS, '26)

Let μ be a singular measure on $\mathbb{T}$. If

$$\omega_\mu(t) \leq C \frac{t}{\sqrt{\log \frac{e}{t}}},$$

then S_μ is cyclic in ℓ_A^p for all $p > 2$. In particular, $S_\mu \notin \mathcal{M}_p$.

For all $p > 2$,

$$B^p := \left\{ f \in \text{Hol}(\mathbb{D}) \mid \int_{\mathbb{D}} |f'(z)|^p (1 - |z|)^{p-1} dA(z) < \infty \right\} \subset \ell_A^p$$

Thus

$$\sup_{0 < t < 1} \left\| \frac{S_\mu(\cdot)}{S_\mu(t \cdot)} \right\|_{B^p} < \infty \implies S_\mu \text{ cyclic in } \ell_A^p$$

Cyclicity in Besov Spaces

Note that

$$\int_{\mathbb{D}} \left| \left(\frac{S_{\mu}(z)}{S_{\mu}(tz)} \right)' \right|^p (1 - |z|)^{p-1} dA(z) \lesssim \int_{\mathbb{D}} \frac{|S'_{\mu}(z)|^p}{|S_{\mu}(tz)|^{2p}} (1 - |z|)^{p-1} dA(z).$$

Suppose $\omega_{\mu}(t) \leq C \frac{t}{\sqrt{\log \frac{e}{t}}}$:

Cyclicity in Besov Spaces

Note that

$$\int_{\mathbb{D}} \left| \left(\frac{S_{\mu}(z)}{S_{\mu}(tz)} \right)' \right|^p (1 - |z|)^{p-1} dA(z) \lesssim \int_{\mathbb{D}} \frac{|S'_{\mu}(z)|^p}{|S_{\mu}(tz)|^{2p}} (1 - |z|)^{p-1} dA(z).$$

Suppose $\omega_{\mu}(t) \leq C \frac{t}{\sqrt{\log \frac{e}{t}}}$:

- **(Anderson, Fernández and Shields, 1991)**

$$\sup_{|z|=r} |S'_{\mu}(z)| \leq C \frac{1}{(1-r)\sqrt{\log \frac{e}{1-r}}} \quad r \in (0, 1)$$

- **(DS, '26)** For all $\varepsilon > 0$, there exists an $\alpha > 0$ such that

$$\int_{\mathbb{T}} \frac{1}{|S_{\mu}(te^{i\theta})|^{\alpha p}} d\theta \leq C \left(\log \frac{e}{1-t} \right)^{\varepsilon}.$$

Concluding Remarks

- Every S_μ is in $\mathcal{M}_2 = H^\infty$;
- If $p \neq 2$:
 - **(Lebedev, '98)** If μ is sufficiently concentrated, then $S_\mu \notin \mathcal{M}_p$
 - **(DS, '26)** If μ is sufficiently "spread out", then $S_\mu \notin \mathcal{M}_p$

Conjecture

*Let $p \neq 2$ and μ be a singular measure on $\mathbb{T}$. Then S_μ is **not** in $\mathcal{M}_p$*

Concluding Remarks

- Every S_μ is in $\mathcal{M}_2 = H^\infty$;
- If $p \neq 2$:
 - (Lebedev, '98) If μ is sufficiently concentrated, then $S_\mu \notin \mathcal{M}_p$
 - (DS, '26) If μ is sufficiently "spread out", then $S_\mu \notin \mathcal{M}_p$

Conjecture

Let $p \neq 2$ and μ be a singular measure on $\mathbb{T}$. Then S_μ is **not** in $\mathcal{M}_p$

Thank you!

Maximal theorems for weighted analytic Tent spaces

José Angel Peláez

Joint work with T. Aguilar-Hernández, A. Mas and J. Rättyä

May 28, 2026

Notation

- $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$;
- $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. $\mathcal{H}(\mathbb{D}) = \{g : g \text{ analytic in } \mathbb{D}\}$.
- For $0 < p \leq \infty$, H^p is the **Hardy space**.
- If $\omega \geq 0$ and $\omega \in L^1$, it is called a weight. If $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$ and $\int_r^1 \omega(s) \in (0, \infty)$, $r \in (0, 1)$, ω is called a radial weight.
- For $\xi \in \overline{\mathbb{D}} \setminus \{0\}$ and $M > 0$,
 $\Gamma_M(\xi) = \{z \in \mathbb{D} : |\arg z - \arg \xi| < M(|\xi| - |z|)\}$. $\Gamma_1(\xi) = \Gamma(\xi)$.
- For $0 < p, q < \infty$ and a weight ω the tent space $T_p^q(\omega)$ is the set of those f for which

$$\|f\|_{T_p^q(\omega)}^q = \int_{\partial \mathbb{D}} \left(\int_{\Gamma(\xi)} |f(z)|^p \omega(z) \frac{dA(z)}{1 - |z|} \right)^{\frac{q}{p}} |d\xi| < \infty.$$

- $AT_p^q(\omega) = T_p^q(\omega) \cap \mathcal{H}(\mathbb{D})$, $T_p^p(\omega) = L_\omega^p$, $AT_p^p(\omega) = A_\omega^p =$ Bergman space induced by ω .

Notation

- $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$;
- $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. $\mathcal{H}(\mathbb{D}) = \{g : g \text{ analytic in } \mathbb{D}\}$.
- For $0 < p \leq \infty$, H^p is the **Hardy space**.
- If $\omega \geq 0$ and $\omega \in L^1$, it is called a weight. If $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$ and $\int_r^1 \omega(s) \in (0, \infty)$, $r \in (0, 1)$, ω is called a radial weight.
- For $\xi \in \overline{\mathbb{D}} \setminus \{0\}$ and $M > 0$,
 $\Gamma_M(\xi) = \{z \in \mathbb{D} : |\arg z - \arg \xi| < M(|\xi| - |z|)\}$. $\Gamma_1(\xi) = \Gamma(\xi)$.
- For $0 < p, q < \infty$ and a weight ω the tent space $T_p^q(\omega)$ is the set of those f for which

$$\|f\|_{T_p^q(\omega)}^q = \int_{\partial \mathbb{D}} \left(\int_{\Gamma(\xi)} |f(z)|^p \omega(z) \frac{dA(z)}{1 - |z|} \right)^{\frac{q}{p}} |d\xi| < \infty.$$

- $AT_p^q(\omega) = T_p^q(\omega) \cap \mathcal{H}(\mathbb{D})$, $T_p^p(\omega) = L_\omega^p$, $AT_p^p(\omega) = A_\omega^p =$ Bergman space induced by ω .

Notation

- $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$;
- $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. $\mathcal{H}(\mathbb{D}) = \{g : g \text{ analytic in } \mathbb{D}\}$.
- For $0 < p \leq \infty$, H^p is the **Hardy space**.
- If $\omega \geq 0$ and $\omega \in L^1$, it is called a weight. If $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$ and $\int_r^1 \omega(s) \in (0, \infty)$, $r \in (0, 1)$, ω is called a radial weight.
- For $\xi \in \overline{\mathbb{D}} \setminus \{0\}$ and $M > 0$,
 $\Gamma_M(\xi) = \{z \in \mathbb{D} : |\arg z - \arg \xi| < M(|\xi| - |z|)\}$. $\Gamma_1(\xi) = \Gamma(\xi)$.
- For $0 < p, q < \infty$ and a weight ω the tent space $T_p^q(\omega)$ is the set of those f for which

$$\|f\|_{T_p^q(\omega)}^q = \int_{\partial \mathbb{D}} \left(\int_{\Gamma(\xi)} |f(z)|^p \omega(z) \frac{dA(z)}{1 - |z|} \right)^{\frac{q}{p}} |d\xi| < \infty.$$

- $AT_p^q(\omega) = T_p^q(\omega) \cap \mathcal{H}(\mathbb{D})$, $T_p^p(\omega) = L_\omega^p$, $AT_p^p(\omega) = A_\omega^p =$ Bergman space induced by ω .

Notation

- $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$;
- $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. $\mathcal{H}(\mathbb{D}) = \{g : g \text{ analytic in } \mathbb{D}\}$.
- For $0 < p \leq \infty$, H^p is the **Hardy space**.
- If $\omega \geq 0$ and $\omega \in L^1$, it is called a weight. If $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$ and $\int_r^1 \omega(s) \in (0, \infty)$, $r \in (0, 1)$, ω is called a radial weight.
- For $\xi \in \overline{\mathbb{D}} \setminus \{0\}$ and $M > 0$,
 $\Gamma_M(\xi) = \{z \in \mathbb{D} : |\arg z - \arg \xi| < M(|\xi| - |z|)\}$. $\Gamma_1(\xi) = \Gamma(\xi)$.
- For $0 < p, q < \infty$ and a weight ω the tent space $T_p^q(\omega)$ is the set of those f for which

$$\|f\|_{T_p^q(\omega)}^q = \int_{\partial \mathbb{D}} \left(\int_{\Gamma(\xi)} |f(z)|^p \omega(z) \frac{dA(z)}{1 - |z|} \right)^{\frac{q}{p}} |d\xi| < \infty.$$

- $AT_p^q(\omega) = T_p^q(\omega) \cap \mathcal{H}(\mathbb{D})$, $T_p^p(\omega) = L_\omega^p$, $AT_p^p(\omega) = A_\omega^p =$ Bergman space induced by ω .

Notation

- $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$;
- $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. $\mathcal{H}(\mathbb{D}) = \{g : g \text{ analytic in } \mathbb{D}\}$.
- For $0 < p \leq \infty$, H^p is the **Hardy space**.
- If $\omega \geq 0$ and $\omega \in L^1$, it is called a weight. If $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$ and $\int_r^1 \omega(s) \in (0, \infty)$, $r \in (0, 1)$, ω is called a radial weight.
- For $\xi \in \overline{\mathbb{D}} \setminus \{0\}$ and $M > 0$,
 $\Gamma_M(\xi) = \{z \in \mathbb{D} : |\arg z - \arg \xi| < M(|\xi| - |z|)\}$. $\Gamma_1(\xi) = \Gamma(\xi)$.
- For $0 < p, q < \infty$ and a weight ω the tent space $T_p^q(\omega)$ is the set of those f for which

$$\|f\|_{T_p^q(\omega)}^q = \int_{\partial \mathbb{D}} \left(\int_{\Gamma(\xi)} |f(z)|^p \omega(z) \frac{dA(z)}{1 - |z|} \right)^{\frac{q}{p}} |d\xi| < \infty.$$

- $AT_p^q(\omega) = T_p^q(\omega) \cap \mathcal{H}(\mathbb{D})$, $T_p^p(\omega) = L_\omega^p$, $AT_p^p(\omega) = A_\omega^p =$ Bergman space induced by ω .

Notation

- $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$;
- $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. $\mathcal{H}(\mathbb{D}) = \{g : g \text{ analytic in } \mathbb{D}\}$.
- For $0 < p \leq \infty$, H^p is the **Hardy space**.
- If $\omega \geq 0$ and $\omega \in L^1$, it is called a weight. If $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$ and $\int_r^1 \omega(s) \in (0, \infty)$, $r \in (0, 1)$, ω is called a radial weight.
- For $\xi \in \overline{\mathbb{D}} \setminus \{0\}$ and $M > 0$,
 $\Gamma_M(\xi) = \{z \in \mathbb{D} : |\arg z - \arg \xi| < M(|\xi| - |z|)\}$. $\Gamma_1(\xi) = \Gamma(\xi)$.
- For $0 < p, q < \infty$ and a weight ω the tent space $T_p^q(\omega)$ is the set of those f for which

$$\|f\|_{T_p^q(\omega)}^q = \int_{\partial \mathbb{D}} \left(\int_{\Gamma(\xi)} |f(z)|^p \omega(z) \frac{dA(z)}{1 - |z|} \right)^{\frac{q}{p}} |d\xi| < \infty.$$

- $AT_p^q(\omega) = T_p^q(\omega) \cap \mathcal{H}(\mathbb{D})$, $T_p^p(\omega) = L_\omega^p$, $AT_p^p(\omega) = A_\omega^p =$ Bergman space induced by ω .

Notation

- $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$;
- $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. $\mathcal{H}(\mathbb{D}) = \{g : g \text{ analytic in } \mathbb{D}\}$.
- For $0 < p \leq \infty$, H^p is the **Hardy space**.
- If $\omega \geq 0$ and $\omega \in L^1$, it is called a weight. If $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$ and $\int_r^1 \omega(s) \in (0, \infty)$, $r \in (0, 1)$, ω is called a radial weight.
- For $\xi \in \overline{\mathbb{D}} \setminus \{0\}$ and $M > 0$,
 $\Gamma_M(\xi) = \{z \in \mathbb{D} : |\arg z - \arg \xi| < M(|\xi| - |z|)\}$. $\Gamma_1(\xi) = \Gamma(\xi)$.
- For $0 < p, q < \infty$ and a weight ω the tent space $T_p^q(\omega)$ is the set of those f for which

$$\|f\|_{T_p^q(\omega)}^q = \int_{\partial \mathbb{D}} \left(\int_{\Gamma(\xi)} |f(z)|^p \omega(z) \frac{dA(z)}{1 - |z|} \right)^{\frac{q}{p}} |d\xi| < \infty.$$

- $AT_p^q(\omega) = T_p^q(\omega) \cap \mathcal{H}(\mathbb{D})$, $T_p^p(\omega) = L_\omega^p$, $AT_p^p(\omega) = A_\omega^p =$ Bergman space induced by ω .

Notation

- $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$;
- $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. $\mathcal{H}(\mathbb{D}) = \{g : g \text{ analytic in } \mathbb{D}\}$.
- For $0 < p \leq \infty$, H^p is the **Hardy space**.
- If $\omega \geq 0$ and $\omega \in L^1$, it is called a weight. If $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$ and $\int_r^1 \omega(s) \in (0, \infty)$, $r \in (0, 1)$, ω is called a radial weight.
- For $\xi \in \overline{\mathbb{D}} \setminus \{0\}$ and $M > 0$,
 $\Gamma_M(\xi) = \{z \in \mathbb{D} : |\arg z - \arg \xi| < M(|\xi| - |z|)\}$. $\Gamma_1(\xi) = \Gamma(\xi)$.
- For $0 < p, q < \infty$ and a weight ω the tent space $T_p^q(\omega)$ is the set of those f for which

$$\|f\|_{T_p^q(\omega)}^q = \int_{\partial \mathbb{D}} \left(\int_{\Gamma(\xi)} |f(z)|^p \omega(z) \frac{dA(z)}{1 - |z|} \right)^{\frac{q}{p}} |d\xi| < \infty.$$

- $AT_p^q(\omega) = T_p^q(\omega) \cap \mathcal{H}(\mathbb{D})$, $T_p^p(\omega) = L_\omega^p$, $AT_p^p(\omega) = A_\omega^p =$ Bergman space induced by ω .

Notation

- $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$;
- $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. $\mathcal{H}(\mathbb{D}) = \{g : g \text{ analytic in } \mathbb{D}\}$.
- For $0 < p \leq \infty$, H^p is the **Hardy space**.
- If $\omega \geq 0$ and $\omega \in L^1$, it is called a weight. If $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$ and $\int_r^1 \omega(s) \in (0, \infty)$, $r \in (0, 1)$, ω is called a radial weight.
- For $\xi \in \overline{\mathbb{D}} \setminus \{0\}$ and $M > 0$,
 $\Gamma_M(\xi) = \{z \in \mathbb{D} : |\arg z - \arg \xi| < M(|\xi| - |z|)\}$. $\Gamma_1(\xi) = \Gamma(\xi)$.
- For $0 < p, q < \infty$ and a weight ω the tent space $T_p^q(\omega)$ is the set of those f for which

$$\|f\|_{T_p^q(\omega)}^q = \int_{\partial \mathbb{D}} \left(\int_{\Gamma(\xi)} |f(z)|^p \omega(z) \frac{dA(z)}{1 - |z|} \right)^{\frac{q}{p}} |d\xi| < \infty.$$

- $AT_p^q(\omega) = T_p^q(\omega) \cap \mathcal{H}(\mathbb{D})$, $T_p^p(\omega) = L_\omega^p$, $AT_p^p(\omega) = A_\omega^p =$ Bergman space induced by ω .

Notation

- $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$;
- $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. $\mathcal{H}(\mathbb{D}) = \{g : g \text{ analytic in } \mathbb{D}\}$.
- For $0 < p \leq \infty$, H^p is the **Hardy space**.
- If $\omega \geq 0$ and $\omega \in L^1$, it is called a weight. If $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$ and $\int_r^1 \omega(s) \in (0, \infty)$, $r \in (0, 1)$, ω is called a radial weight.
- For $\xi \in \overline{\mathbb{D}} \setminus \{0\}$ and $M > 0$,
 $\Gamma_M(\xi) = \{z \in \mathbb{D} : |\arg z - \arg \xi| < M(|\xi| - |z|)\}$. $\Gamma_1(\xi) = \Gamma(\xi)$.
- For $0 < p, q < \infty$ and a weight ω the tent space $T_p^q(\omega)$ is the set of those f for which

$$\|f\|_{T_p^q(\omega)}^q = \int_{\partial\mathbb{D}} \left(\int_{\Gamma(\xi)} |f(z)|^p \omega(z) \frac{dA(z)}{1 - |z|} \right)^{\frac{q}{p}} |d\xi| < \infty.$$

- $AT_p^q(\omega) = T_p^q(\omega) \cap \mathcal{H}(\mathbb{D})$, $T_p^p(\omega) = L_\omega^p$, $AT_p^p(\omega) = A_\omega^p =$ Bergman space induced by ω .

Notation

- $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$;
- $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. $\mathcal{H}(\mathbb{D}) = \{g : g \text{ analytic in } \mathbb{D}\}$.
- For $0 < p \leq \infty$, H^p is the **Hardy space**.
- If $\omega \geq 0$ and $\omega \in L^1$, it is called a weight. If $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$ and $\int_r^1 \omega(s) \in (0, \infty)$, $r \in (0, 1)$, ω is called a radial weight.
- For $\xi \in \overline{\mathbb{D}} \setminus \{0\}$ and $M > 0$,
 $\Gamma_M(\xi) = \{z \in \mathbb{D} : |\arg z - \arg \xi| < M(|\xi| - |z|)\}$. $\Gamma_1(\xi) = \Gamma(\xi)$.
- For $0 < p, q < \infty$ and a weight ω the tent space $T_p^q(\omega)$ is the set of those f for which

$$\|f\|_{T_p^q(\omega)}^q = \int_{\partial \mathbb{D}} \left(\int_{\Gamma(\xi)} |f(z)|^p \omega(z) \frac{dA(z)}{1 - |z|} \right)^{\frac{q}{p}} |d\xi| < \infty.$$

- $AT_p^q(\omega) = T_p^q(\omega) \cap \mathcal{H}(\mathbb{D})$, $T_p^p(\omega) = L_\omega^p$, $AT_p^p(\omega) = A_\omega^p =$ Bergman space induced by ω .

Notation

- $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$;
- $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. $\mathcal{H}(\mathbb{D}) = \{g : g \text{ analytic in } \mathbb{D}\}$.
- For $0 < p \leq \infty$, H^p is the **Hardy space**.
- If $\omega \geq 0$ and $\omega \in L^1$, it is called a weight. If $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$ and $\int_r^1 \omega(s) \in (0, \infty)$, $r \in (0, 1)$, ω is called a radial weight.
- For $\xi \in \overline{\mathbb{D}} \setminus \{0\}$ and $M > 0$,
 $\Gamma_M(\xi) = \{z \in \mathbb{D} : |\arg z - \arg \xi| < M(|\xi| - |z|)\}$. $\Gamma_1(\xi) = \Gamma(\xi)$.
- For $0 < p, q < \infty$ and a weight ω the tent space $T_p^q(\omega)$ is the set of those f for which

$$\|f\|_{T_p^q(\omega)}^q = \int_{\partial \mathbb{D}} \left(\int_{\Gamma(\xi)} |f(z)|^p \omega(z) \frac{dA(z)}{1 - |z|} \right)^{\frac{q}{p}} |d\xi| < \infty.$$

- $AT_p^q(\omega) = T_p^q(\omega) \cap \mathcal{H}(\mathbb{D})$, $T_p^p(\omega) = L_\omega^p$, $AT_p^p(\omega) = A_\omega^p$ = Bergman space induced by ω .

Notation

- $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$;
- $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. $\mathcal{H}(\mathbb{D}) = \{g : g \text{ analytic in } \mathbb{D}\}$.
- For $0 < p \leq \infty$, H^p is the **Hardy space**.
- If $\omega \geq 0$ and $\omega \in L^1$, it is called a weight. If $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$ and $\int_r^1 \omega(s) \in (0, \infty)$, $r \in (0, 1)$, ω is called a radial weight.
- For $\xi \in \overline{\mathbb{D}} \setminus \{0\}$ and $M > 0$,
 $\Gamma_M(\xi) = \{z \in \mathbb{D} : |\arg z - \arg \xi| < M(|\xi| - |z|)\}$. $\Gamma_1(\xi) = \Gamma(\xi)$.
- For $0 < p, q < \infty$ and a weight ω the tent space $T_p^q(\omega)$ is the set of those f for which

$$\|f\|_{T_p^q(\omega)}^q = \int_{\partial \mathbb{D}} \left(\int_{\Gamma(\xi)} |f(z)|^p \omega(z) \frac{dA(z)}{1 - |z|} \right)^{\frac{q}{p}} |d\xi| < \infty.$$

- $AT_p^q(\omega) = T_p^q(\omega) \cap \mathcal{H}(\mathbb{D})$, $T_p^p(\omega) = L_\omega^p$, $AT_p^p(\omega) = A_\omega^p =$ Bergman space induced by ω .

Background on Tent spaces

Why are relevant Tent spaces?

Because of their close connection with the theory of Hardy spaces:

Calderón Formula:

$$\begin{aligned}\|f\|_{H^q}^q &\asymp |f(0)|^q + \int_{\mathbb{T}} \left(\int_{\Gamma(\xi)} |f'(z)|^2 dA(z) \right)^{\frac{q}{2}} |d\xi| \\ &= |f(0)|^q + \|f'\|_{AT_2^q(1-|z|)}^q, \quad 0 < q < \infty.\end{aligned}$$

Boundedness of the non-tangential maximal function:

$$\|f\|_{H^q}^q \asymp \|N_M(f)\|_{L^q(\mathbb{T})}^q, \quad N(f)(\xi) = \sup_{z \in \Gamma_M(\xi)} |f(z)|, \quad 0 < q < \infty.$$

On the theory of Tent spaces

- The theory of Tent spaces was initiated by Coifman-Meyer-Stein (1985); Harboure-Torrea-Viviani (1991-96); Luecking (1991), Cohn-Verbitsky (2000).
- On the theory of analytic weighted Tent spaces for standard weights $\omega(z) = (1 - |z|)^\alpha$, $\alpha > -1$. : Perälä (2018), Aguilar-Hernández-Contreras-R. Piazza (2022).
- Very little is known on analytic weighted Tent spaces induced by radial weights.

Background on Tent spaces

Why are relevant Tent spaces?

Because of their close connection with the theory of Hardy spaces:

Calderón Formula:

$$\begin{aligned}\|f\|_{H^q}^q &\asymp |f(0)|^q + \int_{\mathbb{T}} \left(\int_{\Gamma(\xi)} |f'(z)|^2 dA(z) \right)^{\frac{q}{2}} |d\xi| \\ &= |f(0)|^q + \|f'\|_{AT_2^q(1-|z|)}^q, \quad 0 < q < \infty.\end{aligned}$$

Boundedness of the non-tangential maximal function:

$$\|f\|_{H^q}^q \asymp \|N_M(f)\|_{L^q(\mathbb{T})}^q, \quad N(f)(\xi) = \sup_{z \in \Gamma_M(\xi)} |f(z)|, \quad 0 < q < \infty.$$

On the theory of Tent spaces

- The theory of Tent spaces was initiated by Coifman-Meyer-Stein (1985); Harboure-Torrea-Viviani (1991-96); Luecking (1991), Cohn-Verbitsky (2000).
- On the theory of analytic weighted Tent spaces for standard weights $\omega(z) = (1 - |z|)^\alpha$, $\alpha > -1$: Perälä (2018), Aguilar-Hernández-Contreras-R. Piazza (2022).
- Very little is known on analytic weighted Tent spaces induced by radial weights.

Background on Tent spaces

Why are relevant Tent spaces?

Because of their close connection with the theory of Hardy spaces:

Calderón Formula:

$$\begin{aligned}\|f\|_{H^q}^q &\asymp |f(0)|^q + \int_{\mathbb{T}} \left(\int_{\Gamma(\xi)} |f'(z)|^2 dA(z) \right)^{\frac{q}{2}} |d\xi| \\ &= |f(0)|^q + \|f'\|_{AT_2^q(1-|z|)}^q, \quad 0 < q < \infty.\end{aligned}$$

Boundedness of the non-tangential maximal function:

$$\|f\|_{H^q}^q \asymp \|N_M(f)\|_{L^q(\mathbb{T})}^q, \quad N(f)(\xi) = \sup_{z \in \Gamma_M(\xi)} |f(z)|, \quad 0 < q < \infty.$$

On the theory of Tent spaces

- The theory of Tent spaces was initiated by Coifman-Meyer-Stein (1985); Harboure-Torrea-Viviani (1991-96); Luecking (1991), Cohn-Verbitsky (2000).
- On the theory of analytic weighted Tent spaces for standard weights $\omega(z) = (1 - |z|)^\alpha$, $\alpha > -1$. : Perälä (2018), Aguilar-Hernández-Contreras-R. Piazza (2022).
- Very little is known on analytic weighted Tent spaces induced by radial weights.

Background on Tent spaces

Why are relevant Tent spaces?

Because of their close connection with the theory of Hardy spaces:

Calderón Formula:

$$\begin{aligned}\|f\|_{H^q}^q &\asymp |f(0)|^q + \int_{\mathbb{T}} \left(\int_{\Gamma(\xi)} |f'(z)|^2 dA(z) \right)^{\frac{q}{2}} |d\xi| \\ &= |f(0)|^q + \|f'\|_{AT_2^q(1-|z|)}^q, \quad 0 < q < \infty.\end{aligned}$$

Boundedness of the non-tangential maximal function:

$$\|f\|_{H^q}^q \asymp \|N_M(f)\|_{L^q(\mathbb{T})}^q, \quad N(f)(\xi) = \sup_{z \in \Gamma_M(\xi)} |f(z)|, \quad 0 < q < \infty.$$

On the theory of Tent spaces

- The theory of Tent spaces was initiated by Coifman-Meyer-Stein (1985); Harboure-Torrea-Viviani (1991-96); Luecking (1991), Cohn-Verbitsky (2000).
- On the theory of analytic weighted Tent spaces for standard weights $\omega(z) = (1 - |z|)^\alpha$, $\alpha > -1$. : Perälä (2018), Aguilar-Hernández-Contreras-R. Piazza (2022).
- Very little is known on analytic weighted Tent spaces induced by radial weights.

Background on Tent spaces

Why are relevant Tent spaces?

Because of their close connection with the theory of Hardy spaces:

Calderón Formula:

$$\begin{aligned}\|f\|_{H^q}^q &\asymp |f(0)|^q + \int_{\mathbb{T}} \left(\int_{\Gamma(\xi)} |f'(z)|^2 dA(z) \right)^{\frac{q}{2}} |d\xi| \\ &= |f(0)|^q + \|f'\|_{AT_2^q(1-|z|)}^q, \quad 0 < q < \infty.\end{aligned}$$

Boundedness of the non-tangential maximal function:

$$\|f\|_{H^q}^q \asymp \|N_M(f)\|_{L^q(\mathbb{T})}^q, \quad N(f)(\xi) = \sup_{z \in \Gamma_M(\xi)} |f(z)|, \quad 0 < q < \infty.$$

On the theory of Tent spaces

- The theory of Tent spaces was initiated by Coifman-Meyer-Stein (1985); Harboure-Torrea-Viviani (1991-96); Luecking (1991), Cohn-Verbitsky (2000).
- On the theory of analytic weighted Tent spaces for standard weights $\omega(z) = (1 - |z|)^\alpha$, $\alpha > -1$. : Perälä (2018), Aguilar-Hernández-Contreras-R. Piazza (2022).
- Very little is known on analytic weighted Tent spaces induced by radial weights.

Background on Tent spaces

Why are relevant Tent spaces?

Because of their close connection with the theory of Hardy spaces:

Calderón Formula:

$$\begin{aligned}\|f\|_{H^q}^q &\asymp |f(0)|^q + \int_{\mathbb{T}} \left(\int_{\Gamma(\xi)} |f'(z)|^2 dA(z) \right)^{\frac{q}{2}} |d\xi| \\ &= |f(0)|^q + \|f'\|_{AT_2^q(1-|z|)}^q, \quad 0 < q < \infty.\end{aligned}$$

Boundedness of the non-tangential maximal function:

$$\|f\|_{H^q}^q \asymp \|N_M(f)\|_{L^q(\mathbb{T})}^q, \quad N(f)(\xi) = \sup_{z \in \Gamma_M(\xi)} |f(z)|, \quad 0 < q < \infty.$$

On the theory of Tent spaces

- The theory of Tent spaces was initiated by Coifman-Meyer-Stein (1985); Harboure-Torrea-Viviani (1991-96); Luecking (1991), Cohn-Verbitsky (2000).
- On the theory of analytic weighted Tent spaces for standard weights $\omega(z) = (1 - |z|)^\alpha$, $\alpha > -1$. : Perälä (2018), Aguilar-Hernández-Contreras-R. Piazza (2022).
- Very little is known on analytic weighted Tent spaces induced by radial weights.

Background on Tent spaces

Why are relevant Tent spaces?

Because of their close connection with the theory of Hardy spaces:

Calderón Formula:

$$\begin{aligned}\|f\|_{H^q}^q &\asymp |f(0)|^q + \int_{\mathbb{T}} \left(\int_{\Gamma(\xi)} |f'(z)|^2 dA(z) \right)^{\frac{q}{2}} |d\xi| \\ &= |f(0)|^q + \|f'\|_{AT_2^q(1-|z|)}^q, \quad 0 < q < \infty.\end{aligned}$$

Boundedness of the non-tangential maximal function:

$$\|f\|_{H^q}^q \asymp \|N_M(f)\|_{L^q(\mathbb{T})}^q, \quad N(f)(\xi) = \sup_{z \in \Gamma_M(\xi)} |f(z)|, \quad 0 < q < \infty.$$

On the theory of Tent spaces

- The theory of Tent spaces was initiated by Coifman-Meyer-Stein (1985); Harboure-Torrea-Viviani (1991-96); Luecking (1991), Cohn-Verbitsky (2000).
- On the theory of analytic weighted Tent spaces for standard weights $\omega(z) = (1 - |z|)^\alpha$, $\alpha > -1$. : Perälä (2018), Aguilar-Hernández-Contreras-R. Piazza (2022).
- Very little is known on analytic weighted Tent spaces induced by radial weights.

Background on Tent spaces

Why are relevant Tent spaces?

Because of their close connection with the theory of Hardy spaces:

Calderón Formula:

$$\begin{aligned}\|f\|_{H^q}^q &\asymp |f(0)|^q + \int_{\mathbb{T}} \left(\int_{\Gamma(\xi)} |f'(z)|^2 dA(z) \right)^{\frac{q}{2}} |d\xi| \\ &= |f(0)|^q + \|f'\|_{AT_2^q(1-|z|)}^q, \quad 0 < q < \infty.\end{aligned}$$

Boundedness of the non-tangential maximal function:

$$\|f\|_{H^q}^q \asymp \|N_M(f)\|_{L^q(\mathbb{T})}^q, \quad N(f)(\xi) = \sup_{z \in \Gamma_M(\xi)} |f(z)|, \quad 0 < q < \infty.$$

On the theory of Tent spaces

- The theory of Tent spaces was initiated by Coifman-Meyer-Stein (1985); Harboure-Torrea-Viviani (1991-96); Luecking (1991), Cohn-Verbitsky (2000).
- On the theory of analytic weighted Tent spaces for standard weights $\omega(z) = (1 - |z|)^\alpha$, $\alpha > -1$. : Perälä (2018), Aguilar-Hernández-Contreras-R. Piazza (2022).
- Very little is known on analytic weighted Tent spaces induced by radial weights.

Some aims of the work

- There has been significant progress in operator (and function) theory on weighted Bergman spaces A_ω^p over the last decades. A key difference lies on the nature of the weight; in particular, the theory is far more developed for radial weights.
- **Approximation results:** If ω is a radial weight, polynomials are dense in A_ω^p , $0 < p < \infty$, and if $f_\lambda(z) = f(\lambda z)$, $\lambda \in \overline{\mathbb{D}}$, then:

$$\|f_\lambda\|_{A_\omega^p} \leq \|f\|_{A_\omega^p}, \quad \lambda \in \overline{\mathbb{D}};$$
$$\lim_{\lambda \rightarrow \xi} \|f_\lambda - f_\xi\|_{A_\omega^p} = 0, \quad \xi \in \mathbb{T}, f \in A_\omega^p.$$

Do these results remain true for $AT_p^q(\omega)$?

- The answer is positive for $\omega = 1$:
Aguilar-Hernández-Contreras-R. Piazza (2022).
- It seems that the employed method does not work for a general radial weight ω .

Some aims of the work

- There has been significant progress in operator (and function) theory on weighted Bergman spaces A_ω^p over the last decades. A key difference lies on the nature of the weight; in particular, the theory is far more developed for radial weights.
- **Approximation results:** If ω is a radial weight, polynomials are dense in A_ω^p , $0 < p < \infty$, and if $f_\lambda(z) = f(\lambda z)$, $\lambda \in \overline{\mathbb{D}}$, then:

$$\begin{aligned} \|f_\lambda\|_{A_\omega^p} &\leq \|f\|_{A_\omega^p}, \quad \lambda \in \overline{\mathbb{D}}; \\ \lim_{\lambda \rightarrow \xi} \|f_\lambda - f_\xi\|_{A_\omega^p} &= 0, \quad \xi \in \mathbb{T}, f \in A_\omega^p. \end{aligned}$$

Do these results remain true for $AT_p^q(\omega)$?

- The answer is positive for $\omega = 1$:
Aguilar-Hernández-Contreras-R. Piazza (2022).
- It seems that the employed method does not work for a general radial weight ω .

Some aims of the work

- There has been significant progress in operator (and function) theory on weighted Bergman spaces A_ω^p over the last decades. A key difference lies on the nature of the weight; in particular, the theory is far more developed for radial weights.
- **Approximation results:** If ω is a radial weight, polynomials are dense in A_ω^p , $0 < p < \infty$, and if $f_\lambda(z) = f(\lambda z)$, $\lambda \in \overline{\mathbb{D}}$, then:

$$\begin{aligned} \|f_\lambda\|_{A_\omega^p} &\leq \|f\|_{A_\omega^p}, \quad \lambda \in \overline{\mathbb{D}}; \\ \lim_{\lambda \rightarrow \xi} \|f_\lambda - f_\xi\|_{A_\omega^p} &= 0, \quad \xi \in \mathbb{T}, f \in A_\omega^p. \end{aligned}$$

Do these results remain true for $AT_p^q(\omega)$?

- The answer is positive for $\omega = 1$:
Aguilar-Hernández-Contreras-R. Piazza (2022).
- It seems that the employed method does not work for a general radial weight ω .

Some aims of the work

- There has been significant progress in operator (and function) theory on weighted Bergman spaces A_ω^p over the last decades. A key difference lies on the nature of the weight; in particular, the theory is far more developed for radial weights.
- **Approximation results:** If ω is a radial weight, polynomials are dense in A_ω^p , $0 < p < \infty$, and if $f_\lambda(z) = f(\lambda z)$, $\lambda \in \overline{\mathbb{D}}$, then:

$$\|f_\lambda\|_{A_\omega^p} \leq \|f\|_{A_\omega^p}, \quad \lambda \in \overline{\mathbb{D}};$$

$$\lim_{\lambda \rightarrow \xi} \|f_\lambda - f_\xi\|_{A_\omega^p} = 0, \quad \xi \in \mathbb{T}, f \in A_\omega^p.$$

Do these results remain true for $AT_p^q(\omega)$?

- The answer is positive for $\omega = 1$:
Aguilar-Hernández-Contreras-R. Piazza (2022).
- It seems that the employed method does not work for a general radial weight ω .

Some aims of the work

- There has been significant progress in operator (and function) theory on weighted Bergman spaces A_ω^p over the last decades. A key difference lies on the nature of the weight; in particular, the theory is far more developed for radial weights.
- **Approximation results:** If ω is a radial weight, polynomials are dense in A_ω^p , $0 < p < \infty$, and if $f_\lambda(z) = f(\lambda z)$, $\lambda \in \overline{\mathbb{D}}$, then:

$$\|f_\lambda\|_{A_\omega^p} \leq \|f\|_{A_\omega^p}, \quad \lambda \in \overline{\mathbb{D}};$$

$$\lim_{\lambda \rightarrow \xi} \|f_\lambda - f_\xi\|_{A_\omega^p} = 0, \quad \xi \in \mathbb{T}, f \in A_\omega^p.$$

Do these results remain true for $AT_p^q(\omega)$?

- The answer is positive for $\omega = 1$:
Aguilar-Hernández-Contreras-R. Piazza (2022).
- It seems that the employed method does not work for a general radial weight ω .

Some aims of the work

- There has been significant progress in operator (and function) theory on weighted Bergman spaces A_ω^p over the last decades. A key difference lies on the nature of the weight; in particular, the theory is far more developed for radial weights.
- **Approximation results:** If ω is a radial weight, polynomials are dense in A_ω^p , $0 < p < \infty$, and if $f_\lambda(z) = f(\lambda z)$, $\lambda \in \overline{\mathbb{D}}$, then:

$$\|f_\lambda\|_{A_\omega^p} \leq \|f\|_{A_\omega^p}, \quad \lambda \in \overline{\mathbb{D}};$$
$$\lim_{\lambda \rightarrow \xi} \|f_\lambda - f_\xi\|_{A_\omega^p} = 0, \quad \xi \in \mathbb{T}, f \in A_\omega^p.$$

Do these results remain true for $AT_p^q(\omega)$?

- The answer is positive for $\omega = 1$:
Aguilar-Hernández-Contreras-R. Piazza (2022).
- It seems that the employed method does not work for a general radial weight ω .

Some aims of the work

- There has been significant progress in operator (and function) theory on weighted Bergman spaces A_ω^p over the last decades. A key difference lies on the nature of the weight; in particular, the theory is far more developed for radial weights.
- **Approximation results:** If ω is a radial weight, polynomials are dense in A_ω^p , $0 < p < \infty$, and if $f_\lambda(z) = f(\lambda z)$, $\lambda \in \overline{\mathbb{D}}$, then:

$$\|f_\lambda\|_{A_\omega^p} \leq \|f\|_{A_\omega^p}, \quad \lambda \in \overline{\mathbb{D}};$$
$$\lim_{\lambda \rightarrow \xi} \|f_\lambda - f_\xi\|_{A_\omega^p} = 0, \quad \xi \in \mathbb{T}, f \in A_\omega^p.$$

Do these results remain true for $AT_p^q(\omega)$?

- The answer is positive for $\omega = 1$:
Aguilar-Hernández-Contreras-R. Piazza (2022).
- It seems that the employed method does not work for a general radial weight ω .

Some aims of the work

- There has been significant progress in operator (and function) theory on weighted Bergman spaces A_ω^p over the last decades. A key difference lies on the nature of the weight; in particular, the theory is far more developed for radial weights.
- **Approximation results:** If ω is a radial weight, polynomials are dense in A_ω^p , $0 < p < \infty$, and if $f_\lambda(z) = f(\lambda z)$, $\lambda \in \overline{\mathbb{D}}$, then:

$$\|f_\lambda\|_{A_\omega^p} \leq \|f\|_{A_\omega^p}, \quad \lambda \in \overline{\mathbb{D}};$$

$$\lim_{\lambda \rightarrow \xi} \|f_\lambda - f_\xi\|_{A_\omega^p} = 0, \quad \xi \in \mathbb{T}, f \in A_\omega^p.$$

Do these results remain true for $AT_p^q(\omega)$?

- The answer is positive for $\omega = 1$:
Aguilar-Hernández-Contreras-R. Piazza (2022).
- It seems that the employed method does not work for a general radial weight ω .

Some aims of the work

- There has been significant progress in operator (and function) theory on weighted Bergman spaces A_ω^p over the last decades. A key difference lies on the nature of the weight; in particular, the theory is far more developed for radial weights.
- **Approximation results:** If ω is a radial weight, polynomials are dense in A_ω^p , $0 < p < \infty$, and if $f_\lambda(z) = f(\lambda z)$, $\lambda \in \overline{\mathbb{D}}$, then:

$$\|f_\lambda\|_{A_\omega^p} \leq \|f\|_{A_\omega^p}, \quad \lambda \in \overline{\mathbb{D}};$$

$$\lim_{\lambda \rightarrow \xi} \|f_\lambda - f_\xi\|_{A_\omega^p} = 0, \quad \xi \in \mathbb{T}, f \in A_\omega^p.$$

Do these results remain true for $AT_p^q(\omega)$?

- The answer is positive for $\omega = 1$:
Aguilar-Hernández-Contreras-R. Piazza (2022).
- It seems that the employed method does not work for a general radial weight ω .

Boundedness of the Non-tangential maximal function on weighted analytic Tent spaces

Theorem. Aguilar-Mas-P-Rättyä (2026)

Let $0 < p, q, M < \infty$ and let ω be a radial weight. Then, $N_M : AT_p^q(\omega) \rightarrow T_p^q(\omega)$ is bounded.

Key facts of the proof

1. Fefferman-Stein vectorial maximal theorem,

$$\int_{\mathbb{T}} \left(\sum_{n=0}^{\infty} N_M(f_n)(\xi)^p \right)^{q/p} |d\xi| \lesssim \int_{\mathbb{T}} \left(\sum_{n=0}^{\infty} f_n(\xi)^p \right)^{q/p} |d\xi|$$

valid for $0 < p, q < \infty$ and $\{f_n\}$, $f_n \geq 0$, log-subharmonic functions on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$.

2. $\Gamma_M(\xi) = \bigcup_{z \in \Gamma_M(\xi)} \Gamma_M(z)$, $\xi \in \overline{\mathbb{D}} \setminus \{0\}$.
3. For each $n \in \mathbb{N}$, decompose $\Gamma_M(\xi) = \bigcup_{j=0}^{n-1} \bigcup_{-j \leq k \leq j} E_{j,k}^n(\xi)$ where $E_{j,k}^n$ are disjoint sets and $E_{j,k}^n \subset \Gamma_M(\xi \frac{n-j}{n} e^{iM|\xi|\frac{k}{n}})$.

Boundedness of the Non-tangential maximal function on weighted analytic Tent spaces

Theorem. Aguilar-Mas-P-Rättyä (2026)

Let $0 < p, q, M < \infty$ and let ω be a radial weight. Then, $N_M : AT_p^q(\omega) \rightarrow T_p^q(\omega)$ is bounded.

Key facts of the proof

1. Fefferman-Stein vectorial maximal theorem,

$$\int_{\mathbb{T}} \left(\sum_{n=0}^{\infty} N_M(f_n)(\xi)^p \right)^{q/p} |d\xi| \lesssim \int_{\mathbb{T}} \left(\sum_{n=0}^{\infty} f_n(\xi)^p \right)^{q/p} |d\xi|$$

valid for $0 < p, q < \infty$ and $\{f_n\}$, $f_n \geq 0$, log-subharmonic functions on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$.

2. $\Gamma_M(\xi) = \bigcup_{z \in \Gamma_M(\xi)} \Gamma_M(z)$, $\xi \in \overline{\mathbb{D}} \setminus \{0\}$.
3. For each $n \in \mathbb{N}$, decompose $\Gamma_M(\xi) = \bigcup_{j=0}^{n-1} \bigcup_{-j \leq k \leq j} E_{j,k}^n(\xi)$ where $E_{j,k}^n$ are disjoint sets and $E_{j,k}^n \subset \Gamma_M(\xi \frac{n-j}{n} e^{iM|\xi|\frac{k}{n}})$.

Decomposition of a non-tangential region $\Gamma_M(\xi)$

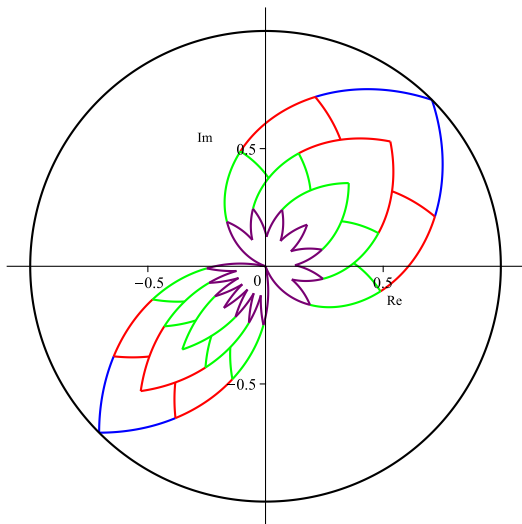


Figure 1: Two real scale illustrations of the sets $E_{j,k}^n(\xi)$.

Approximation by dilated functions and density of polynomials on analytic Tent spaces

Theorem. Aguilar-Mas-P-Rättyä (2026)

Let $0 < p, q < \infty$ and let ω be a radial weight. Then the following statements hold:

(i) There exists a constant $C = C(p, q) > 0$ such that

$$\|f_\lambda\|_{AT_p^q(\omega)} \leq C \|f\|_{AT_p^q(\omega)}, \quad \lambda \in \overline{\mathbb{D}}, \quad f \in \mathcal{H}(\mathbb{D});$$

(ii) $\lim_{\lambda \rightarrow \zeta, \lambda \in \overline{\mathbb{D}}} \|f_\lambda - f_\zeta\|_{AT_p^q(\omega)} = 0$ for each $f \in AT_p^q(\omega)$ and $\zeta \in \overline{\mathbb{D}}$;

(iii) Polynomials are dense in $AT_p^q(\omega)$.

Weighted average radial integrability space.

- Let ω be a radial weight and $0 < p, q < \infty$, $L_p^q(\omega)$ consists of functions f such that

$$\|f\|_{L_p^q(\omega)}^q = \frac{1}{2\pi} \int_0^{2\pi} \left(\int_0^1 |f(re^{i\theta})|^p \omega(r) r dr \right)^{\frac{q}{p}} d\theta < \infty.$$

- $L^{p,p}(\omega) = L_\omega^p$, $AL_p^q(\omega) = L_p^q(\omega) \cap \mathcal{H}(\mathbb{D})$, $AL^{p,p}(\omega) = A_\omega^p$.

Why are relevant these spaces?

- They are also close related to the theory of Hardy spaces:
- Littlewood-Paley formula.

$$\|f\|_{H^p}^p \asymp |f(0)|^p + \int_0^{2\pi} \left(\int_0^1 |f'(re^{i\theta})|^2 (1-r) dr \right)^{p/2} d\theta,$$

- Boundedness of the maximal radial function

$$R(\phi)(z) = \sup_{0 \leq r \leq 1} |\phi(rz)|, \quad z \in \overline{\mathbb{D}}.$$

$$\|f\|_{H^p}^p \asymp \|R(f)\|_{L^p(\mathbb{T})}^p, \quad 0 < p < \infty.$$

Weighted average radial integrability space.

- Let ω be a radial weight and $0 < p, q < \infty$, $L_p^q(\omega)$ consists of functions f such that

$$\|f\|_{L_p^q(\omega)}^q = \frac{1}{2\pi} \int_0^{2\pi} \left(\int_0^1 |f(re^{i\theta})|^p \omega(r) r dr \right)^{\frac{q}{p}} d\theta < \infty.$$

- $L^{p,p}(\omega) = L_\omega^p$, $AL_p^q(\omega) = L_p^q(\omega) \cap \mathcal{H}(\mathbb{D})$, $AL^{p,p}(\omega) = A_\omega^p$.

Why are relevant these spaces?

- They are also close related to the theory of Hardy spaces:
- Littlewood-Paley formula.

$$\|f\|_{H^p}^p \asymp |f(0)|^p + \int_0^{2\pi} \left(\int_0^1 |f'(re^{i\theta})|^2 (1-r) dr \right)^{p/2} d\theta,$$

- Boundedness of the maximal radial function

$$R(\phi)(z) = \sup_{0 \leq r \leq 1} |\phi(rz)|, \quad z \in \overline{\mathbb{D}}.$$

$$\|f\|_{H^p}^p \asymp \|R(f)\|_{L^p(\mathbb{T})}^p, \quad 0 < p < \infty.$$

Weighted average radial integrability space.

- Let ω be a radial weight and $0 < p, q < \infty$, $L_p^q(\omega)$ consists of functions f such that

$$\|f\|_{L_p^q(\omega)}^q = \frac{1}{2\pi} \int_0^{2\pi} \left(\int_0^1 |f(re^{i\theta})|^p \omega(r) r dr \right)^{\frac{q}{p}} d\theta < \infty.$$

- $L^{p,p}(\omega) = L_\omega^p$, $AL_p^q(\omega) = L_p^q(\omega) \cap \mathcal{H}(\mathbb{D})$, $AL^{p,p}(\omega) = A_\omega^p$.

Why are relevant these spaces?

- They are also close related to the theory of Hardy spaces:
- Littlewood-Paley formula.

$$\|f\|_{H^p}^p \asymp |f(0)|^p + \int_0^{2\pi} \left(\int_0^1 |f'(re^{i\theta})|^2 (1-r) dr \right)^{p/2} d\theta,$$

- Boundedness of the maximal radial function

$$R(\phi)(z) = \sup_{0 \leq r \leq 1} |\phi(rz)|, \quad z \in \overline{\mathbb{D}}.$$

$$\|f\|_{H^p}^p \asymp \|R(f)\|_{L^p(\mathbb{T})}^p, \quad 0 < p < \infty.$$

Weighted average radial integrability space.

- Let ω be a radial weight and $0 < p, q < \infty$, $L_p^q(\omega)$ consists of functions f such that

$$\|f\|_{L_p^q(\omega)}^q = \frac{1}{2\pi} \int_0^{2\pi} \left(\int_0^1 |f(re^{i\theta})|^p \omega(r) r dr \right)^{\frac{q}{p}} d\theta < \infty.$$

- $L^{p,p}(\omega) = L_\omega^p$, $AL_p^q(\omega) = L_p^q(\omega) \cap \mathcal{H}(\mathbb{D})$, $AL^{p,p}(\omega) = A_\omega^p$.

Why are relevant these spaces?

- They are also close related to the theory of Hardy spaces:
- Littlewood-Paley formula.

$$\|f\|_{H^p}^p \asymp |f(0)|^p + \int_0^{2\pi} \left(\int_0^1 |f'(re^{i\theta})|^2 (1-r) dr \right)^{p/2} d\theta,$$

- Boundedness of the maximal radial function

$$R(\phi)(z) = \sup_{0 \leq r \leq 1} |\phi(rz)|, \quad z \in \overline{\mathbb{D}}.$$

$$\|f\|_{H^p}^p \asymp \|R(f)\|_{L^p(\mathbb{T})}^p, \quad 0 < p < \infty.$$

On the theory of analytic weighted average radial integrability space

- Analytic weighted average radial integrability space induced by $\omega(z) = (1 - |z|)^\alpha$, $\alpha > -1$, are close related with the Triebel-Lizorkin spaces, Ortega-Fábrega (1997);
- Extensive study for $\omega = 1$: Aguilar-Hernández-Contreras-R. Piazza (2022).

Some new results on analytic weighted average radial integrability space

- Aguilar-Mas-P-Rättyä (2026): Let $0 < p, q, M < \infty$ and let ω be a radial weight. Then, $N_M : AL_p^q(\omega) \rightarrow L_p^q(\omega)$ is bounded.
- Approximation by dilated functions and polynomials work in $AL_p^q(\omega)$, for $0 < p, q < \infty$ and ω a radial weight.

On the theory of analytic weighted average radial integrability space

- Analytic weighted average radial integrability space induced by $\omega(z) = (1 - |z|)^\alpha$, $\alpha > -1$, are close related with the Triebel-Lizorkin spaces, Ortega-Fábrega (1997);
- Extensive study for $\omega = 1$: Aguilar-Hernández-Contreras-R. Piazza (2022).

Some new results on analytic weighted average radial integrability space

- Aguilar-Mas-P-Rättyä (2026): Let $0 < p, q, M < \infty$ and let ω be a radial weight. Then, $N_M : AL_p^q(\omega) \rightarrow L_p^q(\omega)$ is bounded.
- Approximation by dilated functions and polynomials work in $AL_p^q(\omega)$, for $0 < p, q < \infty$ and ω a radial weight.

On the theory of analytic weighted average radial integrability space

- Analytic weighted average radial integrability space induced by $\omega(z) = (1 - |z|)^\alpha$, $\alpha > -1$, are close related with the Triebel-Lizorkin spaces, Ortega-Fábrega (1997);
- Extensive study for $\omega = 1$: Aguilar-Hernández-Contreras-R. Piazza (2022).

Some new results on analytic weighted average radial integrability space

- Aguilar-Mas-P-Rättyä (2026): Let $0 < p, q, M < \infty$ and let ω be a radial weight. Then, $N_M : AL_p^q(\omega) \rightarrow L_p^q(\omega)$ is bounded.
- Approximation by dilated functions and polynomials work in $AL_p^q(\omega)$, for $0 < p, q < \infty$ and ω a radial weight.

On the theory of analytic weighted average radial integrability space

- Analytic weighted average radial integrability space induced by $\omega(z) = (1 - |z|)^\alpha$, $\alpha > -1$, are close related with the Triebel-Lizorkin spaces, Ortega-Fábrega (1997);
- Extensive study for $\omega = 1$:
Aguilar-Hernández-Contreras-R. Piazza (2022).

Some new results on analytic weighted average radial integrability space

- Aguilar-Mas-P-Rättyä (2026): Let $0 < p, q, M < \infty$ and let ω be a radial weight. Then, $N_M : AL_p^q(\omega) \rightarrow L_p^q(\omega)$ is bounded.
- Approximation by dilated functions and polynomials work in $AL_p^q(\omega)$, for $0 < p, q < \infty$ and ω a radial weight.

Weighted analytic Tent spaces versus Weighted average radial integrability spaces.

- Ortega-Fábrega (1997): If $\omega(z) = (1 - |z|)^\alpha$, $\alpha > -1$ and $1 \leq p, q < \infty$. Then,

$$AL_p^q(\omega) = AT_p^q(\omega).$$

Does remain true this result for any radial weight?

- No, but it is true for a meaningful large class of radial weights.

Weighted analytic Tent spaces versus Weighted average radial integrability spaces.

- Ortega-Fábrega (1997): If $\omega(z) = (1 - |z|)^\alpha$, $\alpha > -1$ and $1 \leq p, q < \infty$. Then,

$$AL_p^q(\omega) = AT_p^q(\omega).$$

Does remain true this result for any radial weight?

- No, but it is true for a meaningful large class of radial weights.

Weighted analytic Tent spaces versus Weighted average radial integrability spaces.

- Ortega-Fábrega (1997): If $\omega(z) = (1 - |z|)^\alpha$, $\alpha > -1$ and $1 \leq p, q < \infty$. Then,

$$AL_p^q(\omega) = AT_p^q(\omega).$$

Does remain true this result for any radial weight?

- No, but it is true for a meaningful large class of radial weights.

Weighted analytic Tent spaces versus Weighted average radial integrability spaces.

- Ortega-Fábrega (1997): If $\omega(z) = (1 - |z|)^\alpha$, $\alpha > -1$ and $1 \leq p, q < \infty$. Then,

$$AL_p^q(\omega) = AT_p^q(\omega).$$

Does remain true this result for any radial weight?

- No, but it is true for a meaningful large class of radial weights.

Theorem. Aguilar-Mas-P-Rättyä (2026)

Let $0 < p, q < \infty$ and let ω be a radial weight such that

$$\sup_{0 \leq r < 1} \frac{\int_r^1 \omega(s) ds}{\int_{\frac{1+r}{2}}^1 \omega(s) ds} < \infty. \quad (1)$$

Then, $AL_p^q(\omega) = AT_p^q(\omega)$.

Comments on the above result

- The above theorem does not remain true for a large class of weights which contains $\omega(r) = \exp\left(-\frac{\alpha}{(1-r^l)^\beta}\right)$, $0 < \alpha, l, \beta < \infty$.
- The boundedness of $N_M : AL_p^q(\omega) \rightarrow L_p^q(\omega)$ and $N_M : T_p^q(\omega) \rightarrow T_p^q(\omega)$ are key tools in the proof.
- The class of radial weights which satisfy (1) is denoted by $\widehat{\mathcal{D}}$.

Theorem. Aguilar-Mas-P-Rättyä (2026)

Let $0 < p, q < \infty$ and let ω be a radial weight such that

$$\sup_{0 \leq r < 1} \frac{\int_r^1 \omega(s) ds}{\int_{\frac{1+r}{2}}^1 \omega(s) ds} < \infty. \quad (1)$$

Then, $AL_p^q(\omega) = AT_p^q(\omega)$.

Comments on the above result

- The above theorem does not remain true for a large class of weights which contains $\omega(r) = \exp\left(-\frac{\alpha}{(1-r^l)^\beta}\right)$, $0 < \alpha, l, \beta < \infty$.
- The boundedness of $N_M : AL_p^q(\omega) \rightarrow L_p^q(\omega)$ and $N_M : T_p^q(\omega) \rightarrow T_p^q(\omega)$ are key tools in the proof.
- The class of radial weights which satisfy (1) is denoted by $\widehat{\mathcal{D}}$.

Theorem. Aguilar-Mas-P-Rättyä (2026)

Let $0 < p, q < \infty$ and let ω be a radial weight such that

$$\sup_{0 \leq r < 1} \frac{\int_r^1 \omega(s) ds}{\int_{\frac{1+r}{2}}^1 \omega(s) ds} < \infty. \quad (1)$$

Then, $AL_p^q(\omega) = AT_p^q(\omega)$.

Comments on the above result

- The above theorem does not remain true for a large class of weights which contains $\omega(r) = \exp\left(-\frac{\alpha}{(1-r^l)^\beta}\right)$, $0 < \alpha, l, \beta < \infty$.
- The boundedness of $N_M : AL_p^q(\omega) \rightarrow L_p^q(\omega)$ and $N_M : T_p^q(\omega) \rightarrow T_p^q(\omega)$ are key tools in the proof.
- The class of radial weights which satisfy (1) is denoted by $\widehat{\mathcal{D}}$.

Theorem. Aguilar-Mas-P-Rättyä (2026)

Let $0 < p, q < \infty$ and let ω be a radial weight such that

$$\sup_{0 \leq r < 1} \frac{\int_r^1 \omega(s) ds}{\int_{\frac{1+r}{2}}^1 \omega(s) ds} < \infty. \quad (1)$$

Then, $AL_p^q(\omega) = AT_p^q(\omega)$.

Comments on the above result

- The above theorem does not remain true for a large class of weights which contains $\omega(r) = \exp\left(-\frac{\alpha}{(1-r^l)^\beta}\right)$, $0 < \alpha, l, \beta < \infty$.
- The boundedness of $N_M : AL_p^q(\omega) \rightarrow L_p^q(\omega)$ and $N_M : T_p^q(\omega) \rightarrow T_p^q(\omega)$ are key tools in the proof.
- The class of radial weights which satisfy (1) is denoted by $\widehat{\mathcal{D}}$.

Why it is useful the identity $AL_p^q(\omega) = AT_p^q(\omega)$, $\omega \in \widehat{\mathcal{D}}$?

- To obtain equivalent norms in terms of f' .
- Let be $\mathcal{D}$ the class radial weights weights such that $\omega \in \widehat{\mathcal{D}}$ and there is $K = K(\omega) > 1$ such that $\sup_{0 \leq r < 1} \frac{\int_r^1 \omega(s) ds}{\int_r^{1-\frac{1-r}{K}} \omega(s) ds} < \infty$.

Theorem. Aguilar-Mas-P-Rättyä (2026)

Let $0 < p, q < \infty$ and $k \in \mathbb{N}$, and let ω be a radial weight. Then the following statements are equivalent:

- (i) $\omega \in \mathcal{D}$;
- (ii) $\|f\|_{AT_p^q(\omega)} \asymp \|f^{(k)}(1 - |\cdot|)^k\|_{T_p^q(\omega)} + \sum_{j=0}^{k-1} |f^{(j)}(0)|$, $f \in \mathcal{H}(\mathbb{D})$;
- (iii) $\|f\|_{AL_p^q(\omega)} \asymp \|f^{(k)}(1 - |\cdot|)^k\|_{L_p^q(\omega)} + \sum_{j=0}^{k-1} |f^{(j)}(0)|$, $f \in \mathcal{H}(\mathbb{D})$.

Why it is useful the identity $AL_p^q(\omega) = AT_p^q(\omega)$, $\omega \in \widehat{\mathcal{D}}$?

- To obtain equivalent norms in terms of f' .
- Let be $\mathcal{D}$ the class radial weights weights such that $\omega \in \widehat{\mathcal{D}}$ and there is $K = K(\omega) > 1$ such that $\sup_{0 \leq r < 1} \frac{\int_r^1 \omega(s) ds}{\int_r^{1-\frac{1-r}{K}} \omega(s) ds} < \infty$.

Theorem. Aguilar-Mas-P-Rättyä (2026)

Let $0 < p, q < \infty$ and $k \in \mathbb{N}$, and let ω be a radial weight. Then the following statements are equivalent:

- (i) $\omega \in \mathcal{D}$;
- (ii) $\|f\|_{AT_p^q(\omega)} \asymp \|f^{(k)}(1 - |\cdot|)^k\|_{T_p^q(\omega)} + \sum_{j=0}^{k-1} |f^{(j)}(0)|$, $f \in \mathcal{H}(\mathbb{D})$;
- (iii) $\|f\|_{AL_p^q(\omega)} \asymp \|f^{(k)}(1 - |\cdot|)^k\|_{L_p^q(\omega)} + \sum_{j=0}^{k-1} |f^{(j)}(0)|$, $f \in \mathcal{H}(\mathbb{D})$.

Why it is useful the identity $AL_p^q(\omega) = AT_p^q(\omega)$, $\omega \in \widehat{\mathcal{D}}$?

- To obtain equivalent norms in terms of f' .
- Let be $\mathcal{D}$ the class radial weights weights such that $\omega \in \widehat{\mathcal{D}}$ and there is $K = K(\omega) > 1$ such that $\sup_{0 \leq r < 1} \frac{\int_r^1 \omega(s) ds}{\int_r^{1-\frac{1-r}{K}} \omega(s) ds} < \infty$.

Theorem. Aguilar-Mas-P-Rättyä (2026)

Let $0 < p, q < \infty$ and $k \in \mathbb{N}$, and let ω be a radial weight. Then the following statements are equivalent:

- (i) $\omega \in \mathcal{D}$;
- (ii) $\|f\|_{AT_p^q(\omega)} \asymp \|f^{(k)}(1 - |\cdot|)^k\|_{T_p^q(\omega)} + \sum_{j=0}^{k-1} |f^{(j)}(0)|$, $f \in \mathcal{H}(\mathbb{D})$;
- (iii) $\|f\|_{AL_p^q(\omega)} \asymp \|f^{(k)}(1 - |\cdot|)^k\|_{L_p^q(\omega)} + \sum_{j=0}^{k-1} |f^{(j)}(0)|$, $f \in \mathcal{H}(\mathbb{D})$.

Why it is useful the identity $AL_p^q(\omega) = AT_p^q(\omega)$, $\omega \in \widehat{\mathcal{D}}$?

- To obtain equivalent norms in terms of f' .
- Let be $\mathcal{D}$ the class radial weights weights such that $\omega \in \widehat{\mathcal{D}}$ and there is $K = K(\omega) > 1$ such that $\sup_{0 \leq r < 1} \frac{\int_r^1 \omega(s) ds}{\int_r^{1-\frac{1-r}{K}} \omega(s) ds} < \infty$.

Theorem. Aguilar-Mas-P-Rättyä (2026)

Let $0 < p, q < \infty$ and $k \in \mathbb{N}$, and let ω be a radial weight. Then the following statements are equivalent:

- (i) $\omega \in \mathcal{D}$;
- (ii) $\|f\|_{AT_p^q(\omega)} \asymp \|f^{(k)}(1 - |\cdot|)^k\|_{T_p^q(\omega)} + \sum_{j=0}^{k-1} |f^{(j)}(0)|$, $f \in \mathcal{H}(\mathbb{D})$;
- (iii) $\|f\|_{AL_p^q(\omega)} \asymp \|f^{(k)}(1 - |\cdot|)^k\|_{L_p^q(\omega)} + \sum_{j=0}^{k-1} |f^{(j)}(0)|$, $f \in \mathcal{H}(\mathbb{D})$.

Lipschitz free boundaries in Alt-Caffarelli type problems

Joan Domingo-Pasarin

Universitat de Barcelona

EARCO 2026

Serrin's overdetermined problem

Introduced by Serrin in 1971, it studies the following PDE

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \partial_\nu u = c & \text{on } \partial\Omega, \end{cases} \quad (1)$$

for a bounded domain $\Omega \subset \mathbb{R}^n$ and $c > 0$.

Serrin's overdetermined problem

Introduced by Serrin in 1971, it studies the following PDE

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \partial_\nu u = c & \text{on } \partial\Omega, \end{cases} \quad (1)$$

for a bounded domain $\Omega \subset \mathbb{R}^n$ and $c > 0$.

Theorem (Serrin '71)

Assume Ω is C^2 and $u \in C^2(\overline{\Omega})$ solves (1). Then, $\Omega = B_R$ for some $R = R(n, c)$ and

$$u(x) = \frac{R^2 - |x|^2}{2n}.$$

Serrin's overdetermined problem

- A natural question arising from Serrin's theorem is whether the regularity assumption on Ω and u can be relaxed.

Serrin's overdetermined problem

- A natural question arising from Serrin's theorem is whether the regularity assumption on Ω and u can be relaxed.
- Partial results in this direction: [Vogel '92], [Parjapat '98].

Serrin's overdetermined problem

- A natural question arising from Serrin's theorem is whether the regularity assumption on Ω and u can be relaxed.
- Partial results in this direction: [Vogel '92], [Parjapat '98].
- Very recently, [Figalli, Zhang '26] proves Serrin's theorem in a wider class of domains with finite perimeter.

Serrin's overdetermined problem

- A natural question arising from Serrin's theorem is whether the regularity assumption on Ω and u can be relaxed.
- Partial results in this direction: [Vogel '92], [Parjapat '98].
- Very recently, [Figalli, Zhang '26] proves Serrin's theorem in a wider class of domains with finite perimeter.
- In [Figalli, Zhang '26] the connection of Serrin's problem with the Alt-Caffarelli problem is highlighted as a possible way to prove Serrin's result for Lipschitz domains.

Poisson kernel

- For a bounded smooth domain $\Omega \subset \mathbb{R}^n$, the Green function G^x with pole at $x \in \Omega$ solves

$$\begin{cases} -\Delta G^x = \delta_x & \text{in } \Omega \\ G^x = 0 & \text{on } \partial\Omega \\ \partial_\nu G^x = P^x & \text{on } \partial\Omega. \end{cases}$$

Poisson kernel

- For a bounded smooth domain $\Omega \subset \mathbb{R}^n$, the Green function G^x with pole at $x \in \Omega$ solves

$$\begin{cases} -\Delta G^x = \delta_x & \text{in } \Omega \\ G^x = 0 & \text{on } \partial\Omega \\ \partial_\nu G^x = P^x & \text{on } \partial\Omega. \end{cases}$$

- By classical regularity theory, Ω smooth implies that the Poisson kernel $P^x = \partial_\nu G^x$ is smooth.

Poisson kernel

- For a bounded smooth domain $\Omega \subset \mathbb{R}^n$, the Green function G^x with pole at $x \in \Omega$ solves

$$\begin{cases} -\Delta G^x = \delta_x & \text{in } \Omega \\ G^x = 0 & \text{on } \partial\Omega \\ \partial_\nu G^x = P^x & \text{on } \partial\Omega. \end{cases}$$

- By classical regularity theory, Ω smooth implies that the Poisson kernel $P^x = \partial_\nu G^x$ is smooth.
- Is the converse true? That is, is it true that

$$P^x \text{ smooth for some } x \in \Omega \stackrel{?}{\implies} \Omega \text{ smooth} \quad (2)$$

- For a bounded smooth domain $\Omega \subset \mathbb{R}^n$, the Green function G^x with pole at $x \in \Omega$ solves

$$\begin{cases} -\Delta G^x = \delta_x & \text{in } \Omega \\ G^x = 0 & \text{on } \partial\Omega \\ \partial_\nu G^x = P^x & \text{on } \partial\Omega. \end{cases}$$

- By classical regularity theory, Ω smooth implies that the Poisson kernel $P^x = \partial_\nu G^x$ is smooth.
- Is the converse true? That is, is it true that

$$P^x \text{ smooth for some } x \in \Omega \stackrel{?}{\implies} \Omega \text{ smooth} \quad (2)$$

- [Lewis, Vogel '91] and [Lewis, Vogel '01] show that there exist C^α domains for any $\alpha \in (0, 1)$ such that P^x is constant.

Poisson kernel

- For a bounded smooth domain $\Omega \subset \mathbb{R}^n$, the Green function G^x with pole at $x \in \Omega$ solves

$$\begin{cases} -\Delta G^x = \delta_x & \text{in } \Omega \\ G^x = 0 & \text{on } \partial\Omega \\ \partial_\nu G^x = P^x & \text{on } \partial\Omega. \end{cases}$$

- By classical regularity theory, Ω smooth implies that the Poisson kernel $P^x = \partial_\nu G^x$ is smooth.
- Is the converse true? That is, is it true that

$$P^x \text{ smooth for some } x \in \Omega \stackrel{?}{\implies} \Omega \text{ smooth} \quad (2)$$

- [Lewis, Vogel '91] and [Lewis, Vogel '01] show that there exist C^α domains for any $\alpha \in (0, 1)$ such that P^x is constant.
- It is then natural to ask if (2) holds when Ω is Lipschitz.

Connection with the Alt-Caffarelli problem

Serrin's problem

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \partial_\nu u = c & \text{on } \partial\Omega. \end{cases}$$

Poisson kernel

$$\begin{cases} -\Delta G^x = \delta_x & \text{in } \Omega \\ G^x = 0 & \text{on } \partial\Omega \\ \partial_\nu G^x = P^x & \text{on } \partial\Omega. \end{cases}$$



$$\begin{cases} -\Delta u = f & \text{in } \Omega \cap B_1 \\ u = 0 & \text{on } \partial\Omega \cap B_1 \\ \partial_\nu u = Q & \text{on } \partial\Omega \cap B_1. \end{cases}$$

Alt-Caffarelli type problem

Alt-Caffarelli problem

- Introduced by Alt and Caffarelli in [Alt-Caffarelli '81].

Alt-Caffarelli problem

- Introduced by Alt and Caffarelli in [Alt-Caffarelli '81].
- Studies for which bounded domains $\Omega \subset \mathbb{R}^n$ (with $0 \in \partial\Omega$) there exists a non-negative solution $u : B_1 \rightarrow \mathbb{R}$ of

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \cap B_1 \\ u = 0 & \text{on } \partial\Omega \cap B_1 \\ \partial_\nu u = 1 & \text{on } \partial\Omega \cap B_1. \end{cases}$$

Alt-Caffarelli problem

- Introduced by Alt and Caffarelli in [Alt-Caffarelli '81].
- Studies for which bounded domains $\Omega \subset \mathbb{R}^n$ (with $0 \in \partial\Omega$) there exists a non-negative solution $u : B_1 \rightarrow \mathbb{R}$ of

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \cap B_1 \\ u = 0 & \text{on } \partial\Omega \cap B_1 \\ \partial_\nu u = 1 & \text{on } \partial\Omega \cap B_1. \end{cases}$$

- Extensive literature focused on energy minimizers and viscosity solutions but not on weak solutions!

[Caffarelli '87], [Caffarelli '88], [Caffarelli '89], [Caffarelli, Jerison, König '04], [De Silva, Jerison '09], [De Silva, Savin '11], [Jerison, Savin '15], [Spolaor, Velichkov '19], [Engelstein, Spolaor, Velichkov '20], [Engelstein, Fernández-Real, Yu '23], [Ferreri, Spolaor, Velichkov '24], [Fernández-Real, Yu '26]

Main result

We study the Alt-Caffarelli type problem with right hand side

$$\begin{cases} -\Delta u = f & \text{in } \Omega \cap B_1 \\ u = 0 & \text{on } \partial\Omega \cap B_1 \\ \partial_\nu u = Q & \text{on } \partial\Omega \cap B_1. \end{cases} \quad (3)$$

Main result

We study the Alt-Caffarelli type problem with right hand side

$$\begin{cases} -\Delta u = f & \text{in } \Omega \cap B_1 \\ u = 0 & \text{on } \partial\Omega \cap B_1 \\ \partial_\nu u = Q & \text{on } \partial\Omega \cap B_1. \end{cases} \quad (3)$$

Theorem (D-P., X. Ros-Oton '26)

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain such that $0 \in \partial\Omega$ and assume u solves (3) in the following weak sense: $u \in H_{\text{loc}}^1(B_1)$, $u \geq 0$, $u = 0$ on $B_1 \setminus \Omega$ and

$$-\int_{B_1} \nabla u \cdot \nabla \eta dx = \int_{\partial\Omega \cap B_1} Q \eta d\mathcal{H}^{n-1} - \int_{\Omega \cap B_1} f \eta dx \quad \forall \eta \in C_c^\infty(B_1).$$

If $f, Q \in C^\infty(B_1)$, $f \geq 0$ and $Q > 0$, then $\partial\Omega \cap B_1$ is locally a smooth surface in B_1 .

Thank you!

Estimaciones en L^p con pesos matriciales para Haar-shifts con medidas no doblantes

Fernando Benito Fernández de la Cigoña

Brown University

Trabajo conjunto con T. Borges, F. D'Emilio, M. Pasquariello y N.
Wagner

29 de mayo de 2026

Encuentros de Análisis Real y Complejo

Funciones de Haar

- Fijamos un retículo diádico $\mathcal{D}$ en $\mathbb{R}$.
- Sea μ una medida en $\mathbb{R}$ tal que $0 < \mu(I) < \infty$ para todo $I \in \mathcal{D}$.

Funciones de Haar

- Fijamos un retículo diádico $\mathcal{D}$ en $\mathbb{R}$.
- Sea μ una medida en $\mathbb{R}$ tal que $0 < \mu(I) < \infty$ para todo $I \in \mathcal{D}$.
- I_- es el hijo izquierdo de I e I_+ el derecho. $\hat{I}$ denota el padre de I .
- La función de Haar asociada al intervalo I es

$$h_I = \sqrt{\frac{\mu(I_-)\mu(I_+)}{\mu(I)}} \left(\frac{\chi_{I_-}}{\mu(I_-)} - \frac{\chi_{I_+}}{\mu(I_+)} \right) =: \sqrt{m(I)} \left(\frac{\chi_{I_-}}{\mu(I_-)} - \frac{\chi_{I_+}}{\mu(I_+)} \right).$$

Funciones de Haar

- Fijamos un retículo diádico $\mathcal{D}$ en $\mathbb{R}$.
- Sea μ una medida en $\mathbb{R}$ tal que $0 < \mu(I) < \infty$ para todo $I \in \mathcal{D}$.
- I_- es el hijo izquierdo de I e I_+ el derecho. $\hat{I}$ denota el padre de I .
- La función de Haar asociada al intervalo I es

$$h_I = \sqrt{\frac{\mu(I_-)\mu(I_+)}{\mu(I)}} \left(\frac{\chi_{I_-}}{\mu(I_-)} - \frac{\chi_{I_+}}{\mu(I_+)} \right) =: \sqrt{m(I)} \left(\frac{\chi_{I_-}}{\mu(I_-)} - \frac{\chi_{I_+}}{\mu(I_+)} \right).$$

- Esta función es la única con soporte en I , módulo signo, constante en I_- e I_+ y tal que $\int h_I d\mu = 0$ y $\|h_I\|_{L^2(\mu)} = 1$.
- $\{h_I\}_{I \in \mathcal{D}}$ es una base ortonormal de $L^2(\mu)$.

Funciones de Haar

- Fijamos un retículo diádico $\mathcal{D}$ en $\mathbb{R}$.
- Sea μ una medida en $\mathbb{R}$ tal que $0 < \mu(I) < \infty$ para todo $I \in \mathcal{D}$.
- I_- es el hijo izquierdo de I e I_+ el derecho. $\hat{I}$ denota el padre de I .
- La función de Haar asociada al intervalo I es

$$h_I = \sqrt{\frac{\mu(I_-)\mu(I_+)}{\mu(I)}} \left(\frac{\chi_{I_-}}{\mu(I_-)} - \frac{\chi_{I_+}}{\mu(I_+)} \right) =: \sqrt{m(I)} \left(\frac{\chi_{I_-}}{\mu(I_-)} - \frac{\chi_{I_+}}{\mu(I_+)} \right).$$

- Esta función es la única con soporte en I , módulo signo, constante en I_- e I_+ y tal que $\int h_I d\mu = 0$ y $\|h_I\|_{L^2(\mu)} = 1$.
- $\{h_I\}_{I \in \mathcal{D}}$ es una base ortonormal de $L^2(\mu)$.
- Podemos considerar sistemas de Haar en $\mathbb{R}^n$ y también prescindir de la hipótesis $\mu(I) > 0$.

Haar shifts con medidas no doblantes

- Dado $I \in \mathcal{D}$ y $s, t \in \mathbb{Z}^+$, $I^{(s)}$ denota el s -ésimo ancestro de I y $\mathcal{D}_t(I)$ sus descendientes de orden t .

Haar shifts con medidas no doblantes

- Dado $I \in \mathcal{D}$ y $s, t \in \mathbb{Z}^+$, $I^{(s)}$ denota el s -ésimo ancestro de I y $\mathcal{D}_t(I)$ sus descendientes de orden t .
- Un Haar shift de complejidad (s, t) es un operador T que actúa en las funciones de Haar de la siguiente manera:

$$Th_I = \sum_{J \in \mathcal{D}_t(I^{(s)})} \alpha_{I,J} h_J,$$

donde $\alpha \in \ell^\infty$.

Haar shifts con medidas no doblantes

- Dado $I \in \mathcal{D}$ y $s, t \in \mathbb{Z}^+$, $I^{(s)}$ denota el s -ésimo ancestro de I y $\mathcal{D}_t(I)$ sus descendientes de orden t .
- Un Haar shift de complejidad (s, t) es un operador T que actúa en las funciones de Haar de la siguiente manera:

$$Th_I = \sum_{J \in \mathcal{D}_t(I^{(s)})} \alpha_{I,J} h_J,$$

donde $\alpha \in \ell^\infty$.

- Decimos que una medida μ en $\mathbb{R}$ es equilibrada si $m(I) \sim m(\hat{I})$.

Teorema (López-Sánchez, Martell, Parcet; 2014)

$\|T\|_{L^1(\mu) \rightarrow L^{1,\infty}(\mu)} < \infty$ y $\|T\|_{L^p(\mu) \rightarrow L^p(\mu)}$ para todo Haar shift T y $1 < p < \infty$ si y solo si μ es equilibrada.

Dominación sparse

- Una familia de cubos $\mathcal{S} \subset \mathcal{D}$ es η -sparse si para todo $Q \in \mathcal{S}$ existe $E(Q) \subset Q$ con $\mu(E(Q)) > \eta\mu(Q)$ y tal que $\{E(Q)\}_{Q \in \mathcal{S}}$ son disjuntos dos a dos.

Dominación sparse

- Una familia de cubos $\mathcal{S} \subset \mathcal{D}$ es η -sparse si para todo $Q \in \mathcal{S}$ existe $E(Q) \subset Q$ con $\mu(E(Q)) > \eta\mu(Q)$ y tal que $\{E(Q)\}_{Q \in \mathcal{S}}$ son disjuntos dos a dos.
- Un operador T tiene dominación sparse si existe η tal que para cada f (en una clase buena de funciones) existe $\mathcal{S}$ η -sparse tal que

$$|Tf(x)| \lesssim \sum_{Q \in \mathcal{S}} \langle |f| \rangle_Q \chi_Q(x) =: \mathcal{A}_{\mathcal{S}} f(x).$$

Dominación sparse

- Una familia de cubos $\mathcal{S} \subset \mathcal{D}$ es η -sparse si para todo $Q \in \mathcal{S}$ existe $E(Q) \subset Q$ con $\mu(E(Q)) > \eta\mu(Q)$ y tal que $\{E(Q)\}_{Q \in \mathcal{S}}$ son disjuntos dos a dos.
- Un operador T tiene dominación sparse si existe η tal que para cada f (en una clase buena de funciones) existe $\mathcal{S}$ η -sparse tal que

$$|Tf(x)| \lesssim \sum_{Q \in \mathcal{S}} \langle |f| \rangle_Q \chi_Q(x) =: \mathcal{A}_{\mathcal{S}} f(x).$$

- La dominación sparse permite obtener estimaciones con pesos. Para ello, basta tener una dominación bilineal, a priori más débil:

$$|\langle Tf, g \rangle| \lesssim \sum_{Q \in \mathcal{S}} \langle f \rangle_I \langle g \rangle_I \mu(I) = \langle \mathcal{A}_{\mathcal{S}} f, g \rangle.$$

Dominación sparse para Haar shifts

- Conde-Alonso, Pipher y Wagner probaron que la transformada de Hilbert diádica, un Haar shift de complejidad $(0, 1)$, no admitía dominación sparse en el sentido usual.

Dominación sparse para Haar shifts

- Conde-Alonso, Pipher y Wagner probaron que la transformada de Hilbert diádica, un Haar shift de complejidad $(0, 1)$, no admitía dominación sparse en el sentido usual.

Teorema (Conde-Alonso, Pipher, Wagner; 2024)

Sea μ una medida equilibrada y sin átomos en $\mathbb{R}$ y sea $N \in \mathbb{N}$. Existe $\eta \in (0, 1)$ tal que para todo Haar shift T de complejidad (s, t) con $s + t \leq N$ y para cada par de funciones f y g acotadas, no negativas y de soporte compacto existe una familia $\mathcal{S} \subset \mathcal{D}$ η -sparse tal que

$$|\langle Tf, g \rangle| \lesssim \langle \mathcal{A}_{\mathcal{S}} f, g \rangle + \sum_{\substack{J, K \in \mathcal{S} \\ 2 < \text{dist}(J, K) \leq N+2 \\ J \cap K = \emptyset}} \langle f \rangle_J \langle g \rangle_K \sqrt{m(J)m(K)}.$$

Pesos matriciales

- Un peso matricial es una función W que toma valores en las matrices semidefinidas positivas $\mathcal{M}_d^+$.

Pesos matriciales

- Un peso matricial es una función W que toma valores en las matrices semidefinidas positivas $\mathcal{M}_d^+$.
- Un peso matricial $W : \mathbb{R}^n \rightarrow \mathcal{M}_d^+$ da lugar a una familia de espacios L^p para funciones $f : \mathbb{R}^n \rightarrow \mathbb{R}^d$, definidos por las normas

$$\|f\|_{L^p(W)}^p := \int_{\mathbb{R}^n} |W^{1/p}(x)f(x)|^p d\mu(x).$$

Pesos matriciales

- Un peso matricial es una función W que toma valores en las matrices semidefinidas positivas $\mathcal{M}_d^+$.
- Un peso matricial $W : \mathbb{R}^n \rightarrow \mathcal{M}_d^+$ da lugar a una familia de espacios L^p para funciones $f : \mathbb{R}^n \rightarrow \mathbb{R}^d$, definidos por las normas

$$\|f\|_{L^p(W)}^p := \int_{\mathbb{R}^n} |W^{1/p}(x)f(x)|^p d\mu(x).$$

- Dado $1 < p < \infty$, decimos que $W \in A_p$ si

$$[W]_{A_p} = \sup_Q \int_Q \left(\int_Q |W(x)^{1/p} W(y)^{-1/p}|^{p'} d\mu(y) \right)^{p/p'} d\mu(x) < \infty.$$

- Esta condición caracteriza la acotación de la transformada de Hilbert, y fue determinada por Treil y Volberg para $p = 2$ y por Volberg para $1 < p < \infty$ (1997).

Dominación por cuerpos convexos

- La dominación sparse no es adecuada para obtener estimaciones con pesos matriciales, porque ignora toda la información geométrica.

Dominación por cuerpos convexos

- La dominación sparse no es adecuada para obtener estimaciones con pesos matriciales, porque ignora toda la información geométrica.
- En 2017, Nazarov, Petermichl, Treil y Volberg introdujeron el análogo correcto de la dominación sparse: la dominación (sparse) por cuerpos convexos.

Dominación por cuerpos convexos

- La dominación sparse no es adecuada para obtener estimaciones con pesos matriciales, porque ignora toda la información geométrica.
- En 2017, Nazarov, Petermichl, Treil y Volberg introdujeron el análogo correcto de la dominación sparse: la dominación (sparse) por cuerpos convexos.
- Dada una función $f : \mathbb{R}^n \rightarrow \mathbb{R}^d$ y un cubo $Q \subset \mathbb{R}^n$, definimos el cuerpo convexo promedio

$$\langle\langle f \rangle\rangle_Q = \{\langle \varphi f \rangle_Q : \varphi : Q \rightarrow \mathbb{R}, \|\varphi\|_{L^\infty} \leq 1\}.$$

Dominación por cuerpos convexos

- La dominación sparse no es adecuada para obtener estimaciones con pesos matriciales, porque ignora toda la información geométrica.
- En 2017, Nazarov, Petermichl, Treil y Volberg introdujeron el análogo correcto de la dominación sparse: la dominación (sparse) por cuerpos convexos.
- Dada una función $f : \mathbb{R}^n \rightarrow \mathbb{R}^d$ y un cubo $Q \subset \mathbb{R}^n$, definimos el cuerpo convexo promedio

$$\langle\langle f \rangle\rangle_Q = \{ \langle \varphi f \rangle_Q : \varphi : Q \rightarrow \mathbb{R}, \|\varphi\|_{L^\infty} \leq 1 \}.$$

- La dominación por cuerpos convexos para un operador T es habitualmente

$$Tf(x) \in C \sum_{Q \in \mathcal{S}} \langle\langle f \rangle\rangle_Q \chi_Q(x).$$

Dominación por cuerpos convexos para Haar shifts

- ¿Podemos obtener estimaciones con pesos matriciales para Haar shifts con medidas no doblantes usando ambas ideas?

Dominación por cuerpos convexos para Haar shifts

- ¿Podemos obtener estimaciones con pesos matriciales para Haar shifts con medidas no doblantes usando ambas ideas?

Teorema (B.C., Borges, D'Emilio, Pasquariello, Wagner; 2026)

Sea μ una medida equilibrada y sin átomos en $\mathbb{R}$. Sea $T : L^2(\mathbb{R}; \mathbb{R}^d) \rightarrow L^2(\mathbb{R}; \mathbb{R}^d)$ un Haar shift de complejidad (s, t) . Existe $\eta \in (0, 1)$ tal que para todo $f \in L^1(\mathbb{R}; \mathbb{R}^d)$ con $\text{supp } f \subset I_0 \in \mathcal{D}$ existe una familia η -sparse $\mathcal{S} \subset \mathcal{D}$ tal que

$$Tf(x) \in C \sum_{I \in \mathcal{S}} \langle\langle f \rangle\rangle_I \chi_I(x) + C \sum_{\substack{J, K \in \mathcal{S} \\ \text{dist}(J, K) \leq s+t+2}} \langle\langle f \rangle\rangle_J \frac{\chi_K(x)}{\mu(K)} \sqrt{m(J)m(K)}$$

en casi todo punto en $x \in Q_0$.

- El resultado también es cierto cuando la dimensión del dominio es $n > 1$.

Estimaciones para el operador sparse

- Las estimaciones en $L^p(W)$ para el operador sparse de cuerpos convexos no son triviales.
- Anteriormente, no existía ningún resultado con medidas no doblantes para $p \neq 2$.

Estimaciones para el operador sparse

- Las estimaciones en $L^p(W)$ para el operador sparse de cuerpos convexos no son triviales.
- Anteriormente, no existía ningún resultado con medidas no doblantes para $p \neq 2$.

Teorema (B.C., Borges, D'Emilio, Pasquariello, Wagner; 2026)

Sea μ una medida de Borel localmente finita en $\mathbb{R}^n$. Sea $\mathcal{S}$ una familia η -sparse. Definimos la forma sparse bilineal

$$\mathcal{A}_{\mathcal{S}}(f, g) = \sum_{Q \in \mathcal{S}} \mu(Q) \int_Q \int_Q |f(x) \cdot g(y)| d\mu(x) d\mu(y).$$

Para $W \in A_p$ se tiene

$$\mathcal{A}_{\mathcal{S}}(W^{-1/p} f, W^{1/p} g) \lesssim [W]_{A_p}^{\frac{p}{p-1} - \frac{1}{p}} \|f\|_{L^p} \|g\|_{L^{p'}}.$$

Estimaciones para el operador sparse

- Esto implica que, si T admite dominación por cuerpos convexos, se tiene

$$\|T\|_{L^p(W) \rightarrow L^p(W)} \lesssim [W]_{A_p}^{\frac{p}{p-1}-1/p}.$$

Estimaciones para el operador sparse

- Esto implica que, si T admite dominación por cuerpos convexos, se tiene

$$\|T\|_{L^p(W) \rightarrow L^p(W)} \lesssim [W]_{A_p}^{\frac{p}{p-1}-1/p}.$$

- Un ejemplo es una transformada de martingalas T_σ , i.e.,

$$T_\sigma f = \sum_{Q \in \mathcal{D}} \sigma_Q \Delta_Q f$$

con $\{\sigma_Q\}_{Q \in \mathcal{D}} \subset \{\pm 1\}$.

Estimaciones para el operador sparse

- Esto implica que, si T admite dominación por cuerpos convexos, se tiene

$$\|T\|_{L^p(W) \rightarrow L^p(W)} \lesssim [W]_{A_p}^{\frac{p}{p-1}-1/p}.$$

- Un ejemplo es una transformada de martingalas T_σ , i.e.,

$$T_\sigma f = \sum_{Q \in \mathcal{D}} \sigma_Q \Delta_Q f$$

con $\{\sigma_Q\}_{Q \in \mathcal{D}} \subset \{\pm 1\}$.

- La dependencia de $[W]_{A_p}$ coincide con la que obtuvieron Cruz-Uribe, Isralowitz y Moen para integrales singulares con medidas doblantes (2018).

Estimaciones para el operador sparse

- Esto implica que, si T admite dominación por cuerpos convexos, se tiene

$$\|T\|_{L^p(W) \rightarrow L^p(W)} \lesssim [W]_{A_p}^{\frac{p}{p-1}-1/p}.$$

- Un ejemplo es una transformada de martingalas T_σ , i.e.,

$$T_\sigma f = \sum_{Q \in \mathcal{D}} \sigma_Q \Delta_Q f$$

con $\{\sigma_Q\}_{Q \in \mathcal{D}} \subset \{\pm 1\}$.

- La dependencia de $[W]_{A_p}$ coincide con la que obtuvieron Cruz-Uribe, Isralowitz y Moen para integrales singulares con medidas doblantes (2018).
- ¿Qué pasa con los Haar shifts, que tienen una forma de dominación más complicada?

Clases de pesos para Haar shifts

- Conde-Alonso, Pipher y Wagner identificaron, en el caso escalar, una nueva clase de pesos w que caracteriza la acotación de Haar shifts en $L^p(w)$.

Clases de pesos para Haar shifts

- Conde-Alonso, Pipher y Wagner identificaron, en el caso escalar, una nueva clase de pesos w que caracteriza la acotación de Haar shifts en $L^p(w)$.
- Para un peso matricial W y $N \in \mathbb{N}$, decimos que $W \in A_p^N$ si

$$[W]_{A_p^N} = \sup c_p(Q, R) \int_Q \left(\int_R |W(x)^{1/p} W(y)^{-1/p}|^{p'} d\mu(y) \right)^{p/p'} d\mu(x),$$

donde el supremo se toma sobre $Q, R \in \mathcal{D}$ tal que $\text{dist}(Q, R) \leq N + 2$ y $c_p(Q, R)$ es una constante de normalización.

Clases de pesos para Haar shifts

- Conde-Alonso, Pipher y Wagner identificaron, en el caso escalar, una nueva clase de pesos w que caracteriza la acotación de Haar shifts en $L^p(w)$.
- Para un peso matricial W y $N \in \mathbb{N}$, decimos que $W \in A_p^N$ si

$$[W]_{A_p^N} = \sup c_p(Q, R) \int_Q \left(\int_R |W(x)^{1/p} W(y)^{-1/p}|^{p'} d\mu(y) \right)^{p/p'} d\mu(x),$$

donde el supremo se toma sobre $Q, R \in \mathcal{D}$ tal que $\text{dist}(Q, R) \leq N + 2$ y $c_p(Q, R)$ es una constante de normalización.

Proposición

Sea $n = 1$. Si los Haar shifts de complejidad (s, t) con $s + t \leq N$ están acotados uniformemente en $L^p(W)$, entonces $W \in A_p^N$.

Estimaciones con pesos para Haar shifts

Teorema (B.C., Borges, D'Emilio, Pasquariello, Wagner; 2026)

Sea $W \in A_p^N$ y sea T un Haar shift de complejidad (s, t) con $s + t = N$. Se tiene

$$\|T\|_{L^p(W) \rightarrow L^p(W)} \lesssim [W]_{A_p}^{\frac{p}{p-1} - \frac{2}{p}} [W]_{A_p^N}^{\frac{1}{p}}.$$

Estimaciones con pesos para Haar shifts

Teorema (B.C., Borges, D'Emilio, Pasquariello, Wagner; 2026)

Sea $W \in A_p^N$ y sea T un Haar shift de complejidad (s, t) con $s + t = N$. Se tiene

$$\|T\|_{L^p(W) \rightarrow L^p(W)} \lesssim [W]_{A_p}^{\frac{p}{p-1} - \frac{2}{p}} [W]_{A_p^N}^{\frac{1}{p}}.$$

- Como en el caso escalar, aunque las estimaciones dependen cuantitativamente de la complejidad del shift, las clases A_p^N son independientes de N .

Estimaciones con pesos para Haar shifts

Teorema (B.C., Borges, D'Emilio, Pasquariello, Wagner; 2026)

Sea $W \in A_p^N$ y sea T un Haar shift de complejidad (s, t) con $s + t = N$. Se tiene

$$\|T\|_{L^p(W) \rightarrow L^p(W)} \lesssim [W]_{A_p^N}^{\frac{p}{p-1} - \frac{2}{p}} [W]_{A_p^N}^{\frac{1}{p}}.$$

- Como en el caso escalar, aunque las estimaciones dependen cuantitativamente de la complejidad del shift, las clases A_p^N son independientes de N .

Proposición

Para $1 < p < \infty$ y $N \in \mathbb{N}$ se tiene

$$[W]_{A_p^b} \leq [W]_{A_p^N} \lesssim \left([W]_{A_p^b}\right)^{2^N - 1}.$$

Una generalización del teorema del embedding de Carleson

- Para obtener estimaciones con pesos matriciales, es habitual utilizar el teorema del embedding de Carleson.

Teorema diádico del embedding de Carleson

$$\sum_{Q \in \mathcal{D}} \langle w^{1/p'} f \rangle_Q^p \alpha_Q \leq C_1 \|f\|_{L^p}^p$$

si y solo si

$$\frac{1}{\mu(P)} \sum_{Q \in \mathcal{D}(P)} \langle w \rangle_Q^p \alpha_Q \leq C_2 \langle w \rangle_P.$$

Una generalización del teorema del embedding de Carleson

Teorema (B.C., Borges, D'Emilio, Pasquariello, Wagner; 2026)

Sea $\{w_Q\}_{Q \in \mathcal{D}}$ una colección de pesos localizados en Q tales que

$$\|w_P w_Q^{-1}\|_{L^\infty(Q)} \lesssim \langle w_P \rangle_Q \langle w_Q \rangle_Q^{-1} \text{ para } Q \subset P.$$

Sea $\{\alpha_Q\}_{Q \in \mathcal{D}} \subset \mathbb{R}$ y $1 < p < \infty$. Entonces

$$\sum_{Q \in \mathcal{D}} \langle w_Q^{1/p'} f \rangle_Q^p \alpha_Q \leq C_1 \|f\|_{L^p}^p \text{ para toda } f \in L^p(\mu)$$

si y solo si

$$\frac{1}{\mu(P)} \sum_{Q \in \mathcal{D}(P)} \langle w_P \rangle_Q \langle w_Q \rangle_Q^{p-1} \alpha_Q \leq C_2 \langle w_P \rangle_P \text{ para todo } P \in \mathcal{D}.$$

Del embedding a estimaciones con pesos matriciales

- La estimación para la forma sparse bilineal con pesos se puede reducir a estimaciones para ciertas funciones cuadrado.
- En concreto, queremos estimar

$$\sum_{Q \in \mathcal{S}} \langle |\mathcal{W}_Q^{-1} W^{1/p}||g| \rangle_Q^{p'} \mu(Q).$$

Del embedding a estimaciones con pesos matriciales

- La estimación para la forma sparse bilinear con pesos se puede reducir a estimaciones para ciertas funciones cuadrado.
- En concreto, queremos estimar

$$\sum_{Q \in \mathcal{S}} \langle |\mathcal{W}_Q^{-1} W^{1/p}| |g| \rangle_Q^{p'} \mu(Q).$$

- Los pesos escalares

$$w_Q = |\mathcal{W}_Q^{-1} W^{1/p}|^p \chi_Q$$

satisfacen la condición de compatibilidad y podemos aplicar el resultado anterior.

Extension theorem for the logarithmic discrete Laplacian

Marta de León Contreras

joint work with Jorge J. Betancor and L. Rodríguez-Mesa

 Extension theorems for logarithmic Schrödinger and discrete Laplacian operators, [arXiv:2604.03638](https://arxiv.org/abs/2604.03638)

The extension problem:

Suppose $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is a smooth bded function and solve the **extension problem**:

$$\begin{cases} (-\Delta)u(x, t) = 0, & (x, t) \in \mathbb{R}^d \times (0, \infty) \\ u(x, 0) = f(x), & x \in \mathbb{R}^d, \end{cases}$$

to obtain a smooth bded function u .

The extension problem:

Suppose $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is a smooth bdd function and solve the **extension problem**:

$$\begin{cases} (-\Delta)u(x, t) = 0, & (x, t) \in \mathbb{R}^d \times (0, \infty) \\ u(x, 0) = f(x), & x \in \mathbb{R}^d, \end{cases}$$

to obtain a smooth bdd function u .

- $-\partial_t u(x, 0) = (-\Delta)^{1/2} f(x).$

The extension problem:

Suppose $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is a smooth bdd function and solve the **extension problem**:

$$\begin{cases} (-\Delta)u(x, t) = 0, & (x, t) \in \mathbb{R}^d \times (0, \infty) \\ u(x, 0) = f(x), & x \in \mathbb{R}^d, \end{cases}$$

to obtain a smooth bdd function u .

- $-\partial_t u(x, 0) = (-\Delta)^{1/2} f(x)$.
- Indeed, we apply twice the operator $T : f \mapsto -\partial_t u(\cdot, 0)$, to obtain $T(T(f))(x) = T(-\partial_t u(x, 0)) = \partial_t^2 u(x, 0) = -\Delta_x f(x)$.

The extension problem:

Suppose $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is a smooth bdd function and solve the **extension problem**:

$$\begin{cases} (-\Delta)u(x, t) = 0, & (x, t) \in \mathbb{R}^d \times (0, \infty) \\ u(x, 0) = f(x), & x \in \mathbb{R}^d, \end{cases}$$

to obtain a smooth bdd function u .

- $-\partial_t u(x, 0) = (-\Delta)^{1/2} f(x)$.
- Indeed, we apply twice the operator $T : f \mapsto -\partial_t u(\cdot, 0)$, to obtain $T(T(f))(x) = T(-\partial_t u(x, 0)) = \partial_t^2 u(x, 0) = -\Delta_x f(x)$.
- Since $T \geq 0$, then $T = (-\Delta)^{1/2}$.

The extension problem:

Suppose $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is a smooth bdd function and solve the **extension problem**:

$$\begin{cases} (-\Delta)u(x, t) = 0, & (x, t) \in \mathbb{R}^d \times (0, \infty) \\ u(x, 0) = f(x), & x \in \mathbb{R}^d, \end{cases}$$

to obtain a smooth bdd function u .

- $-\partial_t u(x, 0) = (-\Delta)^{1/2} f(x)$.
- Indeed, we apply twice the operator $T : f \mapsto -\partial_t u(\cdot, 0)$, to obtain $T(T(f))(x) = T(-\partial_t u(x, 0)) = \partial_t^2 u(x, 0) = -\Delta_x f(x)$.
- Since $T \geq 0$, then $T = (-\Delta)^{1/2}$.
- $(-\Delta)^{1/2} : f \mapsto -\partial_t u(\cdot, 0)$.

The fractional Laplacian

- $\mathcal{F}((-\Delta)f)(\xi) = |\xi|^2 \hat{f}(\xi)$

The fractional Laplacian

- $\mathcal{F}((-\Delta)f)(\xi) = |\xi|^2 \hat{f}(\xi) \implies \mathcal{F}((-\Delta)^\sigma f)(\xi) = |\xi|^{2\sigma} \hat{f}(\xi).$

The fractional Laplacian

- $\mathcal{F}((-\Delta)f)(\xi) = |\xi|^2 \hat{f}(\xi) \implies \mathcal{F}((-\Delta)^\sigma f)(\xi) = |\xi|^{2\sigma} \hat{f}(\xi).$
- $$(-\Delta)^\sigma f(x) = C_{d,\sigma} \text{p.v.} \int_{\mathbb{R}^d} \frac{f(x) - f(y)}{|x - y|^{d+2\sigma}} dy, \quad 0 < \sigma < 1.$$

The fractional Laplacian

- $\mathcal{F}((-\Delta)f)(\xi) = |\xi|^2 \hat{f}(\xi) \implies \mathcal{F}((-\Delta)^\sigma f)(\xi) = |\xi|^{2\sigma} \hat{f}(\xi).$
- $$(-\Delta)^\sigma f(x) = C_{d,\sigma} \text{p.v.} \int_{\mathbb{R}^d} \frac{f(x) - f(y)}{|x - y|^{d+2\sigma}} dy, \quad 0 < \sigma < 1.$$
- $(-\Delta)^\sigma$ is a non-local operator!

The fractional Laplacian

- $\mathcal{F}((-\Delta)f)(\xi) = |\xi|^2 \hat{f}(\xi) \implies \mathcal{F}((-\Delta)^\sigma f)(\xi) = |\xi|^{2\sigma} \hat{f}(\xi).$
- $$(-\Delta)^\sigma f(x) = C_{d,\sigma} \text{p.v.} \int_{\mathbb{R}^d} \frac{f(x) - f(y)}{|x - y|^{d+2\sigma}} dy, \quad 0 < \sigma < 1.$$
- $(-\Delta)^\sigma$ is a non-local operator!
- $f \in L^1_{2\sigma}(\mathbb{R}^d) := \left\{ f : \mathbb{R}^d \rightarrow \mathbb{R} : \int_{\mathbb{R}^d} \frac{|f(x)|}{(1 + |x|)^{d+2\sigma}} dx < \infty \right\}.$

The extension problem:

Theorem (Caffarelli and Silvestre, *Comm. PDE* 2007)

Let $\sigma \in (0, 1)$. For $f \in \mathcal{S}(\mathbb{R}^d)$ we consider the Dirichlet problem

$$\begin{cases} (\Delta_x + \frac{1-2\sigma}{t} \partial_t u + \partial_t^2 u)(x, t) = 0, & (x, t) \in \mathbb{R}^d \times (0, \infty) \\ u(x, 0) = f(x), & x \in \mathbb{R}^d. \end{cases}$$

The extension problem:

Theorem (Caffarelli and Silvestre, *Comm. PDE* 2007)

Let $\sigma \in (0, 1)$. For $f \in \mathcal{S}(\mathbb{R}^d)$ we consider the Dirichlet problem

$$\begin{cases} (\Delta_x + \frac{1-2\sigma}{t} \partial_t u + \partial_t^2 u)(x, t) = 0, & (x, t) \in \mathbb{R}^d \times (0, \infty) \\ u(x, 0) = f(x), & x \in \mathbb{R}^d. \end{cases}$$

Then, the associated Dirichlet to Neumann map coincides with

$$\frac{\Gamma(1-\sigma)}{2^{2\sigma-1}\Gamma(\sigma)}(-\Delta)^\sigma, \text{ i.e., } \frac{\Gamma(1-\sigma)}{2^{2\sigma-1}\Gamma(\sigma)}(-\Delta)^\sigma : f \mapsto -\lim_{t \rightarrow 0^+} t^{1-2\sigma} \partial_t u(\cdot, t).$$

The extension problem:

Theorem (Caffarelli and Silvestre, *Comm. PDE* 2007)

Let $\sigma \in (0, 1)$. For $f \in \mathcal{S}(\mathbb{R}^d)$ we consider the Dirichlet problem

$$\begin{cases} (\Delta_x + \frac{1-2\sigma}{t} \partial_t u + \partial_t^2 u)(x, t) = 0, & (x, t) \in \mathbb{R}^d \times (0, \infty) \\ u(x, 0) = f(x), & x \in \mathbb{R}^d. \end{cases}$$

Then, the associated Dirichlet to Neumann map coincides with

$$\frac{\Gamma(1-\sigma)}{2^{2\sigma-1}\Gamma(\sigma)}(-\Delta)^\sigma, \text{ i.e., } \frac{\Gamma(1-\sigma)}{2^{2\sigma-1}\Gamma(\sigma)}(-\Delta)^\sigma : f \mapsto -\lim_{t \rightarrow 0^+} t^{1-2\sigma} \partial_t u(\cdot, t).$$

- For $\sigma = 1/2$ we recover the extension to the Laplace equation.

The extension problem:

Theorem (Caffarelli and Silvestre, *Comm. PDE* 2007)

Let $\sigma \in (0, 1)$. For $f \in \mathcal{S}(\mathbb{R}^d)$ we consider the Dirichlet problem

$$\begin{cases} (\Delta_x + \frac{1-2\sigma}{t} \partial_t u + \partial_t^2 u)(x, t) = 0, & (x, t) \in \mathbb{R}^d \times (0, \infty) \\ u(x, 0) = f(x), & x \in \mathbb{R}^d. \end{cases}$$

Then, the associated Dirichlet to Neumann map coincides with

$$\frac{\Gamma(1-\sigma)}{2^{2\sigma-1}\Gamma(\sigma)} (-\Delta)^\sigma, \text{ i.e., } \frac{\Gamma(1-\sigma)}{2^{2\sigma-1}\Gamma(\sigma)} (-\Delta)^\sigma : f \mapsto -\lim_{t \rightarrow 0^+} t^{1-2\sigma} \partial_t u(\cdot, t).$$

- For $\sigma = 1/2$ we recover the extension to the Laplace equation.
- The solution to the extension problem is given by

$$u(x, t) = c_{d,\sigma} \int_{\mathbb{R}^d} \frac{t^{2\sigma}}{(|x - y|^2 + t^2)^{\frac{d+2\sigma}{2}}} f(y) dy$$

The extension problem:

Theorem (Caffarelli and Silvestre, *Comm. PDE* 2007)

Let $\sigma \in (0, 1)$. For $f \in \mathcal{S}(\mathbb{R}^d)$ we consider the Dirichlet problem

$$\begin{cases} (\Delta_x + \frac{1-2\sigma}{t} \partial_t u + \partial_t^2 u)(x, t) = 0, & (x, t) \in \mathbb{R}^d \times (0, \infty) \\ u(x, 0) = f(x), & x \in \mathbb{R}^d. \end{cases}$$

Then, the associated Dirichlet to Neumann map coincides with

$$\frac{\Gamma(1-\sigma)}{2^{2\sigma-1}\Gamma(\sigma)}(-\Delta)^\sigma, \text{ i.e., } \frac{\Gamma(1-\sigma)}{2^{2\sigma-1}\Gamma(\sigma)}(-\Delta)^\sigma : f \mapsto -\lim_{t \rightarrow 0^+} t^{1-2\sigma} \partial_t u(\cdot, t).$$

- For $\sigma = 1/2$ we recover the extension to the Laplace equation.
- The solution to the extension problem is given by

$$\begin{aligned} u(x, t) &= c_{d,\sigma} \int_{\mathbb{R}^d} \frac{t^{2\sigma}}{(|x-y|^2 + t^2)^{\frac{d+2\sigma}{2}}} f(y) dy \\ &= \frac{t^{2\sigma}}{4^\sigma \Gamma(\sigma)} \int_0^\infty e^{t\Delta}(f)(x) \frac{e^{-\frac{t^2}{4u}}}{u^{1+\sigma}} du. \end{aligned}$$

Why it is so useful to prove an extension problem?

Why it is so useful to prove an extension problem?

- Characterization of **non-local operators** as a Dirichlet-to-Neumann map for a local degenerate elliptic PDE.

Why it is so useful to prove an extension problem?

- Characterization of **non-local operators** as a Dirichlet-to-Neumann map for a local degenerate elliptic PDE.
- Obtain properties for **non-local operators** by taking advantage of the local PDE techniques applied to the extended equation:

Why it is so useful to prove an extension problem?

- Characterization of **non-local operators** as a Dirichlet-to-Neumann map for a local degenerate elliptic PDE.
- Obtain properties for **non-local operators** by taking advantage of the local PDE techniques applied to the extended equation:
 - Regularity properties of solutions

Why it is so useful to prove an extension problem?

- Characterization of **non-local operators** as a Dirichlet-to-Neumann map for a local degenerate elliptic PDE.
- Obtain properties for **non-local operators** by taking advantage of the local PDE techniques applied to the extended equation:
 - Regularity properties of solutions
 - Harnack inequalities

Why it is so useful to prove an extension problem?

- Characterization of **non-local operators** as a Dirichlet-to-Neumann map for a local degenerate elliptic PDE.
- Obtain properties for **non-local operators** by taking advantage of the local PDE techniques applied to the extended equation:
 - Regularity properties of solutions
 - Harnack inequalities
 - Monotonicity formulas

¿For which operators is valid a representation through an extension problem?

- Stinga and Torrea: L^σ (Comm. PDE 2010), and $(\partial_t - \Delta)^\sigma$ in (SIAM J.M.A, 2017).

¿For which operators is valid a representation through an extension problem?

- Stinga and Torrea: L^σ (Comm. PDE 2010), and $(\partial_t - \Delta)^\sigma$ in (SIAM J.M.A, 2017).
- Biswas, DL-C, Stinga: $(\partial_t + L)^\sigma$ (SIAM J.M.A, 2021).

¿For which operators is valid a representation through an extension problem?

- Stinga and Torrea: L^σ (Comm. PDE 2010), and $(\partial_t - \Delta)^\sigma$ in (SIAM J.M.A, 2017).
- Biswas, DL-C, Stinga: $(\partial_t + L)^\sigma$ (SIAM J.M.A, 2021).
- Galé, Miana and Stinga (J. Evol. Eq., 2013) A^σ (A generator of integrated family of operators).

¿For which operators is valid a representation through an extension problem?

- Stinga and Torrea: L^σ (Comm. PDE 2010), and $(\partial_t - \Delta)^\sigma$ in (SIAM J.M.A, 2017).
- Biswas, DL-C, Stinga: $(\partial_t + L)^\sigma$ (SIAM J.M.A, 2021).
- Galé, Miana and Stinga (J. Evol. Eq., 2013) A^σ (A generator of integrated family of operators).
- Banica, González and Sáez (Rev. Mat. Iberoam., 2015)) and Bhowmik and Pusti (J.F.A., 2022): the fractional Laplace Beltrami operator on some noncompact manifolds.

¿For which operators is valid a representation through an extension problem?

- Stinga and Torrea: L^σ (Comm. PDE 2010), and $(\partial_t - \Delta)^\sigma$ in (SIAM J.M.A, 2017).
- Biswas, DL-C, Stinga: $(\partial_t + L)^\sigma$ (SIAM J.M.A, 2021).
- Galé, Miana and Stinga (J. Evol. Eq., 2013) A^σ (A generator of integrated family of operators).
- Banica, González and Sáez (Rev. Mat. Iberoam., 2015)) and Bhowmik and Pusti (J.F.A., 2022): the fractional Laplace Beltrami operator on some noncompact manifolds.
- Arendt, ter Elst and Warma (Comm. PDE, 2018): sectorial operators in Hilbert spaces.

¿For which operators is valid a representation through an extension problem?

- Stinga and Torrea: L^σ (Comm. PDE 2010), and $(\partial_t - \Delta)^\sigma$ in (SIAM J.M.A, 2017).
- Biswas, DL-C, Stinga: $(\partial_t + L)^\sigma$ (SIAM J.M.A, 2021).
- Galé, Miana and Stinga (J. Evol. Eq., 2013) A^σ (A generator of integrated family of operators).
- Banica, González and Sáez (Rev. Mat. Iberoam., 2015)) and Bhowmik and Pusti (J.F.A., 2022): the fractional Laplace Beltrami operator on some noncompact manifolds.
- Arendt, ter Elst and Warma (Comm. PDE, 2018): sectorial operators in Hilbert spaces.
- Kwaśnicki and Mucha (J. Evol. Eq., 2018): $\phi(-\Delta)$, for ϕ a complete Bernstein function.

¿For which operators is valid a representation through an extension problem?

- Stinga and Torrea: L^σ (Comm. PDE 2010), and $(\partial_t - \Delta)^\sigma$ in (SIAM J.M.A, 2017).
- Biswas, DL-C, Stinga: $(\partial_t + L)^\sigma$ (SIAM J.M.A, 2021).
- Galé, Miana and Stinga (J. Evol. Eq., 2013) A^σ (A generator of integrated family of operators).
- Banica, González and Sáez (Rev. Mat. Iberoam., 2015)) and Bhowmik and Pusti (J.F.A., 2022): the fractional Laplace Beltrami operator on some noncompact manifolds.
- Arendt, ter Elst and Warma (Comm. PDE, 2018): sectorial operators in Hilbert spaces.
- Kwaśnicki and Mucha (J. Evol. Eq., 2018): $\phi(-\Delta)$, for ϕ a complete Bernstein function.
- Chen, Hauer and Weth (2023): for $\log(-\Delta)$ on $\mathbb{R}^d$.

The logarithmic Laplacian

- Chen and Weth (Comm. PDE, 2019) defined $\log(-\Delta)$ as

$$\log(-\Delta)u = \frac{d}{d\sigma}((- \Delta)^\sigma u) \Big|_{\sigma=0}, \quad u \in C_c^\alpha(\mathbb{R}^d).$$

The logarithmic Laplacian

- Chen and Weth (Comm. PDE, 2019) defined $\log(-\Delta)$ as

$$\log(-\Delta)u = \frac{d}{d\sigma}((- \Delta)^\sigma u) \Big|_{\sigma=0}, \quad u \in C_c^\alpha(\mathbb{R}^d).$$

- $\mathcal{F}(\log(-\Delta)u)(\xi) = 2 \log |\xi| \hat{u}(\xi).$

The logarithmic Laplacian

- Chen and Weth (Comm. PDE, 2019) defined $\log(-\Delta)$ as

$$\log(-\Delta)u = \frac{d}{d\sigma}((- \Delta)^\sigma u) \Big|_{\sigma=0}, \quad u \in C_c^\alpha(\mathbb{R}^d).$$

- $\mathcal{F}(\log(-\Delta)u)(\xi) = 2 \log |\xi| \hat{u}(\xi).$
- $\log(-\Delta)u(x) = c_d \text{ p.v. } \int_{\mathbb{R}^d} \frac{u(x)\chi_{B_1(x)} - u(y)}{|x-y|^d} dy + \rho_d u(x).$

The logarithmic Laplacian

- Chen and Weth (Comm. PDE, 2019) defined $\log(-\Delta)$ as

$$\log(-\Delta)u = \frac{d}{d\sigma} ((-\Delta)^\sigma u) \Big|_{\sigma=0}, \quad u \in C_c^\alpha(\mathbb{R}^d).$$

- $\mathcal{F}(\log(-\Delta)u)(\xi) = 2 \log |\xi| \hat{u}(\xi).$
- $\log(-\Delta)u(x) = c_d \text{ p.v. } \int_{\mathbb{R}^d} \frac{u(x)\chi_{B_1(x)} - u(y)}{|x-y|^d} dy + \rho_d u(x).$
- $\log(-\Delta)u(x)$ well-defined for u Dini continuous at $x \in \mathbb{R}^d$, i.e.

$$\int_0^1 \frac{\sup_{y \in B(x,r)} |f(y) - f(x)|}{r} dr < \infty,$$

The logarithmic Laplacian

- Chen and Weth (Comm. PDE, 2019) defined $\log(-\Delta)$ as

$$\log(-\Delta)u = \frac{d}{d\sigma}((- \Delta)^\sigma u) \Big|_{\sigma=0}, \quad u \in C_c^\alpha(\mathbb{R}^d).$$

- $\mathcal{F}(\log(-\Delta)u)(\xi) = 2 \log |\xi| \hat{u}(\xi).$
- $\log(-\Delta)u(x) = c_d \text{ p.v. } \int_{\mathbb{R}^d} \frac{u(x)\chi_{B_1(x)} - u(y)}{|x-y|^d} dy + \rho_d u(x).$
- $\log(-\Delta)u(x)$ well-defined for u Dini continuous at $x \in \mathbb{R}^d$, i.e.

$$\int_0^1 \frac{\sup_{y \in B(x,r)} |f(y) - f(x)|}{r} dr < \infty,$$

$$\text{and } u \in L_0^1(\mathbb{R}^d) := \left\{ f : \mathbb{R}^d \rightarrow \mathbb{R} : \int_{\mathbb{R}^d} \frac{|f(x)|}{(1+|x|)^d} dx < \infty \right\}.$$

The logarithmic Laplacian

- Chen and Weth (Comm. PDE, 2019) defined $\log(-\Delta)$ as

$$\log(-\Delta)u = \frac{d}{d\sigma}((- \Delta)^\sigma u) \Big|_{\sigma=0}, \quad u \in C_c^\alpha(\mathbb{R}^d).$$

- $\mathcal{F}(\log(-\Delta)u)(\xi) = 2 \log |\xi| \hat{u}(\xi).$
- $\log(-\Delta)u(x) = c_d \text{ p.v. } \int_{\mathbb{R}^d} \frac{u(x)\chi_{B_1(x)} - u(y)}{|x-y|^d} dy + \rho_d u(x).$
- $\log(-\Delta)u(x)$ well-defined for u Dini continuous at $x \in \mathbb{R}^d$, i.e.

$$\int_0^1 \frac{\sup_{y \in B(x,r)} |f(y) - f(x)|}{r} dr < \infty,$$

$$\text{and } u \in L_0^1(\mathbb{R}^d) := \left\{ f : \mathbb{R}^d \rightarrow \mathbb{R} : \int_{\mathbb{R}^d} \frac{|f(x)|}{(1+|x|)^d} dx < \infty \right\}.$$

Some applications/problems to study:

Some applications/problems to study:

- The Cauchy problem associated with $\log(-\Delta)$.

Some applications/problems to study:

- The Cauchy problem associated with $\log(-\Delta)$.
- Asymptotics, as $\sigma \rightarrow 0^+$, of the Dirichlet eigenvalues and eigenfunctions of $(-\Delta)^\sigma$ on open bded Lipschitz sets.

Some applications/problems to study:

- The Cauchy problem associated with $\log(-\Delta)$.
- Asymptotics, as $\sigma \rightarrow 0^+$, of the Dirichlet eigenvalues and eigenfunctions of $(-\Delta)^\sigma$ on open bded Lipschitz sets.
 - * $\lambda_1^{(-\Delta)^\sigma} = 1 + \sigma \lambda_1^{\log(-\Delta)} + o(\sigma), \quad \text{as } \sigma \rightarrow 0.$

Some applications/problems to study:

- The Cauchy problem associated with $\log(-\Delta)$.
- Asymptotics, as $\sigma \rightarrow 0^+$, of the Dirichlet eigenvalues and eigenfunctions of $(-\Delta)^\sigma$ on open bded Lipschitz sets.
 - * $\lambda_1^{(-\Delta)^\sigma} = 1 + \sigma \lambda_1^{\log(-\Delta)} + o(\sigma), \quad \text{as } \sigma \rightarrow 0.$
- s -dependent nonlinear Dirichlet problems, limit when $s \rightarrow 0^+$ at its relationship with $\log(-\Delta)$.

Extension for the logarithmic Laplacian

Def: $f \in \mathcal{AG}(\mathbb{R}^d)$ when $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $\exists C, \sigma > 0$:

$$\int_{B(x,1)} |f(y)| dy \leq C(1 + |x|)^\sigma, \quad x \in \mathbb{R}^d.$$

Extension for the logarithmic Laplacian

Theorem (Chen, Hauer, Weth (2023))

For $f \in L^1_0(\mathbb{R}^d)$, the unique solution $u_f \in C^\infty(\mathbb{R}^{d+1}_+) \cap \mathcal{AG}(\mathbb{R}^{d+1}_+)$ to

$$\left(\partial_t^2 + \frac{1}{t}\partial_t + \Delta\right)u_f(x, t) = 0, \quad x \in \mathbb{R}^d \text{ and } t > 0$$

satisfying

(i) $-\lim_{t \rightarrow 0^+} t\partial_t u_f(\cdot, t) = f$ in $L^1_{\text{loc}}(\mathbb{R}^d)$,

(ii) $\lim_{t \rightarrow \infty} u_f(x, t) = 0$, for every $x \in \mathbb{R}^d$,
is given by

$$u_f(x, t) = \frac{1}{2} \int_0^\infty e^{u\Delta}(f)(x) \frac{e^{-\frac{t^2}{4u}}}{u} du, \quad x \in \mathbb{R}^d, \quad t > 0.$$

Extension for the logarithmic Laplacian

Theorem (Chen,Hauer, Weth (2023))

For $f \in L^1_0(\mathbb{R}^d)$, the unique solution $u_f \in C^\infty(\mathbb{R}^{d+1}_+) \cap \mathcal{AG}(\mathbb{R}^{d+1}_+)$ to

$$\left(\partial_t^2 + \frac{1}{t}\partial_t + \Delta\right)u_f(x, t) = 0, \quad x \in \mathbb{R}^d \text{ and } t > 0$$

satisfying

(i) $-\lim_{t \rightarrow 0^+} t\partial_t u_f(\cdot, t) = f$ in $L^1_{\text{loc}}(\mathbb{R}^d)$,

(ii) $\lim_{t \rightarrow \infty} u_f(x, t) = 0$, for every $x \in \mathbb{R}^d$,
is given by

$$u_f(x, t) = \frac{1}{2} \int_0^\infty e^{u\Delta}(f)(x) \frac{e^{-\frac{t^2}{4u}}}{u} du, \quad x \in \mathbb{R}^d, t > 0. \text{ Moreover,}$$

(a) $\log(-\Delta)(f) = -2 \lim_{t \rightarrow 0^+} (u_f(\cdot, t) + f \log t) + (2 \log 2 - \gamma)f$ in the distributional sense, and ptwise wherever f is Dini cont.

Extension for the logarithmic Laplacian

Theorem (Chen,Hauer, Weth (2023))

For $f \in L^1_0(\mathbb{R}^d)$, the unique solution $u_f \in C^\infty(\mathbb{R}^{d+1}_+) \cap \mathcal{AG}(\mathbb{R}^{d+1}_+)$ to

$$(\partial_t^2 + \frac{1}{t}\partial_t + \Delta)u_f(x, t) = 0, \quad x \in \mathbb{R}^d \text{ and } t > 0$$

satisfying

(i) $-\lim_{t \rightarrow 0^+} t \partial_t u_f(\cdot, t) = f$ in $L^1_{\text{loc}}(\mathbb{R}^d)$,

(ii) $\lim_{t \rightarrow \infty} u_f(x, t) = 0$, for every $x \in \mathbb{R}^d$,
is given by

$$u_f(x, t) = \frac{1}{2} \int_0^\infty e^{u\Delta}(f)(x) \frac{e^{-\frac{t^2}{4u}}}{u} du, \quad x \in \mathbb{R}^d, t > 0. \text{ Moreover,}$$

(a) $\log(-\Delta)(f) = -2 \lim_{t \rightarrow 0^+} (u_f(\cdot, t) + f \log t) + (2 \log 2 - \gamma)f$ in the distributional sense, and ptwise wherever f is Dini cont.

(b) $\lim_{t \rightarrow 0^+} \frac{u_f(\cdot, t)}{\log t} = -f$ in $L^1_{\text{loc}}(\mathbb{R}^d)$.

The ideas behind this extension problem:

$$(-\Delta)^\sigma f(x) = -d_\sigma \lim_{t \rightarrow 0^+} t^{1-2\sigma} \partial_t u(x, t), \quad d_\sigma = \frac{2^{2\sigma-1} \Gamma(\sigma)}{\Gamma(1-\sigma)}$$

The ideas behind this extension problem:

$$\begin{aligned}
 (-\Delta)^\sigma f(x) &= -d_\sigma \lim_{t \rightarrow 0^+} t^{1-2\sigma} \partial_t u(x, t), & d_\sigma &= \frac{2^{2\sigma-1} \Gamma(\sigma)}{\Gamma(1-\sigma)} \\
 &= -2\sigma d_\sigma \lim_{t \rightarrow 0^+} \frac{u(x, t) - u(x, 0)}{t^{2\sigma}}
 \end{aligned}$$

The ideas behind this extension problem:

$$\begin{aligned}
 (-\Delta)^\sigma f(x) &= -d_\sigma \lim_{t \rightarrow 0^+} t^{1-2\sigma} \partial_t u(x, t), & d_\sigma &= \frac{2^{2\sigma-1} \Gamma(\sigma)}{\Gamma(1-\sigma)} \\
 &= -2\sigma d_\sigma \lim_{t \rightarrow 0^+} \frac{u(x, t) - u(x, 0)}{t^{2\sigma}} = -2\sigma d_\sigma \lim_{t \rightarrow 0^+} \frac{u(x, t) - f(x)}{t^{2\sigma}}.
 \end{aligned}$$

The ideas behind this extension problem:

$$\begin{aligned}
 (-\Delta)^\sigma f(x) &= -d_\sigma \lim_{t \rightarrow 0^+} t^{1-2\sigma} \partial_t u(x, t), & d_\sigma &= \frac{2^{2\sigma-1} \Gamma(\sigma)}{\Gamma(1-\sigma)} \\
 &= -2\sigma d_\sigma \lim_{t \rightarrow 0^+} \frac{u(x, t) - u(x, 0)}{t^{2\sigma}} = -2\sigma d_\sigma \lim_{t \rightarrow 0^+} \frac{u(x, t) - f(x)}{t^{2\sigma}}.
 \end{aligned}$$

$$\log(-\Delta)f(x) = \lim_{\sigma \rightarrow 0^+} \frac{(-\Delta^\sigma)f(x) - f(x)}{\sigma}$$

The ideas behind this extension problem:

$$\begin{aligned}
 (-\Delta)^\sigma f(x) &= -d_\sigma \lim_{t \rightarrow 0^+} t^{1-2\sigma} \partial_t u(x, t), & d_\sigma &= \frac{2^{2\sigma-1} \Gamma(\sigma)}{\Gamma(1-\sigma)} \\
 &= -2\sigma d_\sigma \lim_{t \rightarrow 0^+} \frac{u(x, t) - u(x, 0)}{t^{2\sigma}} = -2\sigma d_\sigma \lim_{t \rightarrow 0^+} \frac{u(x, t) - f(x)}{t^{2\sigma}}.
 \end{aligned}$$

$$\begin{aligned}
 \log(-\Delta)f(x) &= \lim_{\sigma \rightarrow 0^+} \frac{(-\Delta^\sigma)f(x) - f(x)}{\sigma} \\
 &= - \lim_{\sigma \rightarrow 0^+} \lim_{t \rightarrow 0^+} \left(2d_\sigma \frac{u(x, t) - f(x)}{t^{2\sigma}} + \frac{f(x)}{\sigma} \right)
 \end{aligned}$$

The ideas behind this extension problem:

$$\begin{aligned}
 (-\Delta)^\sigma f(x) &= -d_\sigma \lim_{t \rightarrow 0^+} t^{1-2\sigma} \partial_t u(x, t), & d_\sigma &= \frac{2^{2\sigma-1} \Gamma(\sigma)}{\Gamma(1-\sigma)} \\
 &= -2\sigma d_\sigma \lim_{t \rightarrow 0^+} \frac{u(x, t) - u(x, 0)}{t^{2\sigma}} = -2\sigma d_\sigma \lim_{t \rightarrow 0^+} \frac{u(x, t) - f(x)}{t^{2\sigma}}.
 \end{aligned}$$

$$\begin{aligned}
 \log(-\Delta)f(x) &= \lim_{\sigma \rightarrow 0^+} \frac{(-\Delta^\sigma)f(x) - f(x)}{\sigma} \\
 &= - \lim_{\sigma \rightarrow 0^+} \lim_{t \rightarrow 0^+} \left(2d_\sigma \frac{u(x, t) - f(x)}{t^{2\sigma}} + \frac{f(x)}{\sigma} \right) \\
 &= - \lim_{\sigma \rightarrow 0^+} \lim_{t \rightarrow 0^+} \left(\sigma d_\sigma \frac{2}{\sigma} \frac{u(x, t) - f(x)}{t^{2\sigma}} + 2\sigma d_\sigma \frac{f(x)}{\sigma} - 2\sigma d_\sigma \frac{f(x)}{\sigma} + \frac{f(x)}{\sigma} \right)
 \end{aligned}$$

The ideas behind this extension problem:

$$\begin{aligned}
 (-\Delta)^\sigma f(x) &= -d_\sigma \lim_{t \rightarrow 0^+} t^{1-2\sigma} \partial_t u(x, t), & d_\sigma &= \frac{2^{2\sigma-1} \Gamma(\sigma)}{\Gamma(1-\sigma)} \\
 &= -2\sigma d_\sigma \lim_{t \rightarrow 0^+} \frac{u(x, t) - u(x, 0)}{t^{2\sigma}} = -2\sigma d_\sigma \lim_{t \rightarrow 0^+} \frac{u(x, t) - f(x)}{t^{2\sigma}}.
 \end{aligned}$$

$$\begin{aligned}
 \log(-\Delta)f(x) &= \lim_{\sigma \rightarrow 0^+} \frac{(-\Delta^\sigma)f(x) - f(x)}{\sigma} \\
 &= - \lim_{\sigma \rightarrow 0^+} \lim_{t \rightarrow 0^+} \left(2d_\sigma \frac{u(x, t) - f(x)}{t^{2\sigma}} + \frac{f(x)}{\sigma} \right) \\
 &= - \lim_{\sigma \rightarrow 0^+} \lim_{t \rightarrow 0^+} \left(\sigma d_\sigma \frac{2}{\sigma} \frac{u(x, t) - f(x)}{t^{2\sigma}} + 2\sigma d_\sigma \frac{f(x)}{\sigma} - 2\sigma d_\sigma \frac{f(x)}{\sigma} + \frac{f(x)}{\sigma} \right) \\
 &= - \lim_{\sigma \rightarrow 0^+} \lim_{t \rightarrow 0^+} \left(\sigma d_\sigma v_\sigma(x, t) - \frac{2\sigma d_\sigma - 1}{\sigma} f(x) \right), \\
 & & v_\sigma(x, t) &= \frac{2}{\sigma} \left(\frac{u(x, t) - f(x)}{t^{2\sigma}} + f(x) \right).
 \end{aligned}$$

- $\lim_{\sigma \rightarrow 0^+} \sigma d_\sigma = \frac{1}{2}, \quad \lim_{\sigma \rightarrow 0^+} \frac{2\sigma d_\sigma - 1}{\sigma} = 2(\log 2 - \gamma).$

- $\lim_{\sigma \rightarrow 0^+} \sigma d_\sigma = \frac{1}{2}, \quad \lim_{\sigma \rightarrow 0^+} \frac{2\sigma d_\sigma - 1}{\sigma} = 2(\log 2 - \gamma).$
- $\log(-\Delta)f(x) = -\frac{1}{2} \lim_{\sigma \rightarrow 0^+} \lim_{t \rightarrow 0^+} v_\sigma(x, t) + 2(\log 2 - \gamma)f(x),$

$$v_\sigma(x, t) = \frac{2}{\sigma} \left(\frac{u(x, t) - f(x)}{t^{2\sigma}} + f(x) \right)$$

- $\lim_{\sigma \rightarrow 0^+} \sigma d_\sigma = \frac{1}{2}, \quad \lim_{\sigma \rightarrow 0^+} \frac{2\sigma d_\sigma - 1}{\sigma} = 2(\log 2 - \gamma).$
- $\log(-\Delta)f(x) = -\frac{1}{2} \lim_{\sigma \rightarrow 0^+} \lim_{t \rightarrow 0^+} v_\sigma(x, t) + 2(\log 2 - \gamma)f(x),$

$$v_\sigma(x, t) = \frac{2}{\sigma} \left(\frac{u(x, t) - f(x)}{t^{2\sigma}} + f(x) \right)$$

$$\frac{1}{2} v_\sigma(x, t) = \frac{u(x, t)}{\sigma t^{2\sigma}} - f(x) \left(\frac{t^{-2\sigma}}{\sigma} - \frac{1}{\sigma} \right)$$

- $\lim_{\sigma \rightarrow 0^+} \sigma d_\sigma = \frac{1}{2}, \quad \lim_{\sigma \rightarrow 0^+} \frac{2\sigma d_\sigma - 1}{\sigma} = 2(\log 2 - \gamma).$
- $\log(-\Delta)f(x) = -\frac{1}{2} \lim_{\sigma \rightarrow 0^+} \lim_{t \rightarrow 0^+} v_\sigma(x, t) + 2(\log 2 - \gamma)f(x),$

$$v_\sigma(x, t) = \frac{2}{\sigma} \left(\frac{u(x, t) - f(x)}{t^{2\sigma}} + f(x) \right)$$

$$\begin{aligned} \frac{1}{2} v_\sigma(x, t) &= \frac{u(x, t)}{\sigma t^{2\sigma}} - f(x) \left(\frac{t^{-2\sigma}}{\sigma} - \frac{1}{\sigma} \right) \\ &= \frac{c_{d,\sigma}}{\sigma} \int_{\mathbb{R}^d} \frac{f(y)}{(|x - y|^2 + t^2)^{\frac{d+2\sigma}{2}}} dy - f(x) \left(\frac{t^{-2\sigma} - 1}{\sigma} \right) \end{aligned}$$

- $\lim_{\sigma \rightarrow 0^+} \sigma d_\sigma = \frac{1}{2}, \quad \lim_{\sigma \rightarrow 0^+} \frac{2\sigma d_\sigma - 1}{\sigma} = 2(\log 2 - \gamma).$
- $\log(-\Delta)f(x) = -\frac{1}{2} \lim_{\sigma \rightarrow 0^+} \lim_{t \rightarrow 0^+} v_\sigma(x, t) + 2(\log 2 - \gamma)f(x),$

$$v_\sigma(x, t) = \frac{2}{\sigma} \left(\frac{u(x, t) - f(x)}{t^{2\sigma}} + f(x) \right)$$

$$\begin{aligned} \frac{1}{2} v_\sigma(x, t) &= \frac{u(x, t)}{\sigma t^{2\sigma}} - f(x) \left(\frac{t^{-2\sigma}}{\sigma} - \frac{1}{\sigma} \right) \\ &= \frac{c_{d,\sigma}}{\sigma} \int_{\mathbb{R}^d} \frac{f(y)}{(|x-y|^2 + t^2)^{\frac{d+2\sigma}{2}}} dy - f(x) \left(\frac{t^{-2\sigma} - 1}{\sigma} \right) \\ &\xrightarrow{\sigma \rightarrow 0^+} \frac{\Gamma(d/2)}{\pi^{d/2}} \int_{\mathbb{R}^d} \frac{f(y)}{(|x-y|^2 + t^2)^{\frac{d}{2}}} dy + 2f(x) \log t. \end{aligned}$$

- $\lim_{\sigma \rightarrow 0^+} \sigma d_\sigma = \frac{1}{2}, \quad \lim_{\sigma \rightarrow 0^+} \frac{2\sigma d_\sigma - 1}{\sigma} = 2(\log 2 - \gamma).$
- $\log(-\Delta)f(x) = -\frac{1}{2} \lim_{\sigma \rightarrow 0^+} \lim_{t \rightarrow 0^+} v_\sigma(x, t) + 2(\log 2 - \gamma)f(x),$

$$v_\sigma(x, t) = \frac{2}{\sigma} \left(\frac{u(x, t) - f(x)}{t^{2\sigma}} + f(x) \right)$$

$$\begin{aligned} \frac{1}{2} v_\sigma(x, t) &= \frac{u(x, t)}{\sigma t^{2\sigma}} - f(x) \left(\frac{t^{-2\sigma}}{\sigma} - \frac{1}{\sigma} \right) \\ &= \frac{c_{d,\sigma}}{\sigma} \int_{\mathbb{R}^d} \frac{f(y)}{(|x-y|^2 + t^2)^{\frac{d+2\sigma}{2}}} dy - f(x) \left(\frac{t^{-2\sigma} - 1}{\sigma} \right) \\ &\xrightarrow{\sigma \rightarrow 0^+} \frac{\Gamma(d/2)}{\pi^{d/2}} \int_{\mathbb{R}^d} \frac{f(y)}{(|x-y|^2 + t^2)^{\frac{d}{2}}} dy + 2f(x) \log t. \end{aligned}$$

$$\log(-\Delta)f(x)$$

$$= -2 \lim_{t \rightarrow 0^+} \left(\frac{\Gamma(d/2)}{2\pi^{d/2}} \int_{\mathbb{R}^d} \frac{f(y)}{(|x-y|^2 + t^2)^{\frac{d}{2}}} dy + f(x) \log t \right) + 2(\log 2 - \gamma)f(x).$$

1 Extension for $\log(-\Delta_d)$

The discrete Laplacian

The **discrete Laplacian** on $\mathbb{Z}$, is defined, for $f : \mathbb{Z} \rightarrow \mathbb{R}$, as

$$\Delta_d f(n) := f(n+1) + f(n-1) - 2f(n), \quad n \in \mathbb{Z}.$$

The discrete Laplacian

The **discrete Laplacian** on $\mathbb{Z}$, is defined, for $f : \mathbb{Z} \rightarrow \mathbb{R}$, as

$$\Delta_d f(n) := f(n+1) + f(n-1) - 2f(n), \quad n \in \mathbb{Z}.$$

The solution to

$$\begin{cases} \partial_t u(t, n) = \Delta_d u(t, n), & n \in \mathbb{Z}, t \geq 0, \\ u(0, n) = f(n), & n \in \mathbb{Z}, \end{cases}$$

The discrete Laplacian

The **discrete Laplacian** on $\mathbb{Z}$, is defined, for $f : \mathbb{Z} \rightarrow \mathbb{R}$, as

$$\Delta_d f(n) := f(n+1) + f(n-1) - 2f(n), \quad n \in \mathbb{Z}.$$

The solution to

$$\begin{cases} \partial_t u(t, n) = \Delta_d u(t, n), & n \in \mathbb{Z}, t \geq 0, \\ u(0, n) = f(n), & n \in \mathbb{Z}, \end{cases}$$

is

$$u(t, n) = e^{t\Delta_d} f(n) := \sum_{j \in \mathbb{Z}} p_t(j) f(n-j),$$

where $p_t(n) = e^{-2t} I_n(2t)$, $n \in \mathbb{Z}, t > 0$, and

$$I_n(t) = \sum_{m=0}^{\infty} \frac{1}{m! \Gamma(m+n+1)} \left(\frac{t}{2}\right)^{2m+n}.$$

Definition of $\log(-\Delta_d)$, Chen, Xu (2025)

- $\mathcal{F}((-\Delta_d)f)(\xi) = 4 \sin^2\left(\frac{\xi}{2}\right) \hat{f}(\xi)$

Definition of $\log(-\Delta_d)$, Chen, Xu (2025)

- $\mathcal{F}((-\Delta_d)f)(\xi) = 4 \sin^2\left(\frac{\xi}{2}\right) \hat{f}(\xi)$
 $\implies \mathcal{F}((-\Delta_d)^\sigma f)(\xi) = \left(4 \sin^2\left(\frac{\xi}{2}\right)\right)^\sigma \hat{f}(\xi)$

Definition of $\log(-\Delta_d)$, Chen, Xu (2025)

- $\mathcal{F}((-\Delta_d)f)(\xi) = 4 \sin^2\left(\frac{\xi}{2}\right) \hat{f}(\xi)$
 $\implies \mathcal{F}((-\Delta_d)^\sigma f)(\xi) = \left(4 \sin^2\left(\frac{\xi}{2}\right)\right)^\sigma \hat{f}(\xi)$
 $\implies \mathcal{F}(\log(-\Delta_d)f)(\xi) = \log\left(4 \sin^2\left(\frac{\xi}{2}\right)\right) \hat{f}(\xi), \xi \in \mathbb{T}.$

Definition of $\log(-\Delta_d)$, Chen, Xu (2025)

- $\mathcal{F}((-\Delta_d)f)(\xi) = 4 \sin^2\left(\frac{\xi}{2}\right) \hat{f}(\xi)$
 $\implies \mathcal{F}((-\Delta_d)^\sigma f)(\xi) = \left(4 \sin^2\left(\frac{\xi}{2}\right)\right)^\sigma \hat{f}(\xi)$
 $\implies \mathcal{F}(\log(-\Delta_d)f)(\xi) = \log\left(4 \sin^2\left(\frac{\xi}{2}\right)\right) \hat{f}(\xi), \xi \in \mathbb{T}.$
- From the Frullani integral representation, for $\lambda > 0$,

$$\log(\lambda) = \int_0^\infty \frac{e^{-t} - e^{-\lambda t}}{t} dt.$$

Definition of $\log(-\Delta_d)$, Chen, Xu (2025)

- $\mathcal{F}((-\Delta_d)f)(\xi) = 4 \sin^2\left(\frac{\xi}{2}\right) \hat{f}(\xi)$
 $\implies \mathcal{F}((-\Delta_d)^\sigma f)(\xi) = \left(4 \sin^2\left(\frac{\xi}{2}\right)\right)^\sigma \hat{f}(\xi)$
 $\implies \mathcal{F}(\log(-\Delta_d)f)(\xi) = \log\left(4 \sin^2\left(\frac{\xi}{2}\right)\right) \hat{f}(\xi), \xi \in \mathbb{T}.$
- From the Frullani integral representation, for $\lambda > 0$,

$$\log(\lambda) = \int_0^\infty \frac{e^{-t} - e^{-\lambda t}}{t} dt.$$

This formula motivates the definition

$$\log(-\Delta_d)f = \int_0^\infty \frac{e^{-t}f - e^{t\Delta_d}f}{t} dt.$$

Theorem (Chen, Xu (2025))

Let $f \in C_c(\mathbb{Z})$. We have that

$$\begin{aligned} (\log(-\Delta_d)f)(n) &= \sum_{\substack{m \in \mathbb{Z} \\ m \neq n}} W_0(n-m)(f(n) - f(m)) \\ &\quad - \sum_{m \in \mathbb{Z}} W_\infty(n-m)f(m) - \gamma f(n), \quad n \in \mathbb{Z}, \end{aligned} \quad (1)$$

where the kernels W_0 and W_∞ are given by

$$W_0(m) = \int_0^1 \frac{p_t(m)}{t} dt, \quad W_\infty(m) = \int_1^\infty \frac{p_t(m)}{t} dt, \quad m \in \mathbb{Z}.$$

Here, $p_t(m) = e^{-2t} I_{|m|}(2t)$, $m \in \mathbb{Z}$ and $t > 0$.

- Identity (1) is also valid for $\{f(m)\}_{m \in \mathbb{Z}}$ s.t. $\sum_{m \in \mathbb{Z}} \frac{|f(m)|}{\sqrt{1+|m|}} < \infty$.

Extension problem for $\log(-\Delta_d)$

Theorem (Betancor, DL-C, Rodríguez-Mesa, 2026)

Let $\{f(m)\}_{m \in \mathbb{Z}}$ s.t. $\sum_{m \in \mathbb{Z}} \frac{|f(m)|}{\sqrt{1+|m|}} < \infty$. We define

$$u_f(n, t) = \frac{1}{2} \int_0^\infty e^{u\Delta_d} f(x) \frac{e^{-\frac{t^2}{4u}}}{u} du, \quad n \in \mathbb{Z} \text{ and } t > 0.$$

Extension problem for $\log(-\Delta_d)$

Theorem (Betancor, DL-C, Rodríguez-Mesa, 2026)

Let $\{f(m)\}_{m \in \mathbb{Z}}$ s.t. $\sum_{m \in \mathbb{Z}} \frac{|f(m)|}{\sqrt{1+|m|}} < \infty$. We define

$$u_f(n, t) = \frac{1}{2} \int_0^\infty e^{u\Delta_d} f(x) \frac{e^{-\frac{t^2}{4u}}}{u} du, \quad n \in \mathbb{Z} \text{ and } t > 0.$$

Then,

(i) $(\partial_t^2 + \frac{1}{t}\partial_t + \Delta_d)u_f(n, t) = 0, \quad n \in \mathbb{Z} \text{ and } t > 0.$

Extension problem for $\log(-\Delta_d)$

Theorem (Betancor, DL-C, Rodríguez-Mesa, 2026)

Let $\{f(m)\}_{m \in \mathbb{Z}}$ s.t. $\sum_{m \in \mathbb{Z}} \frac{|f(m)|}{\sqrt{1+|m|}} < \infty$. We define

$$u_f(n, t) = \frac{1}{2} \int_0^\infty e^{u\Delta_d} f(x) \frac{e^{-\frac{t^2}{4u}}}{u} du, \quad n \in \mathbb{Z} \text{ and } t > 0.$$

Then,

- (i) $(\partial_t^2 + \frac{1}{t}\partial_t + \Delta_d)u_f(n, t) = 0, \quad n \in \mathbb{Z} \text{ and } t > 0.$
- (ii) $-\lim_{t \rightarrow 0^+} t\partial_t u_f(n, t) = f(n), \quad n \in \mathbb{Z}.$

Extension problem for $\log(-\Delta_d)$

Theorem (Betancor, DL-C, Rodríguez-Mesa, 2026)

Let $\{f(m)\}_{m \in \mathbb{Z}}$ s.t. $\sum_{m \in \mathbb{Z}} \frac{|f(m)|}{\sqrt{1+|m|}} < \infty$. We define

$$u_f(n, t) = \frac{1}{2} \int_0^\infty e^{u\Delta_d} f(x) \frac{e^{-\frac{t^2}{4u}}}{u} du, \quad n \in \mathbb{Z} \text{ and } t > 0.$$

Then,

- (i) $(\partial_t^2 + \frac{1}{t}\partial_t + \Delta_d)u_f(n, t) = 0, n \in \mathbb{Z} \text{ and } t > 0.$
- (ii) $-\lim_{t \rightarrow 0^+} t\partial_t u_f(n, t) = f(n), n \in \mathbb{Z}.$
- (iii) $\lim_{t \rightarrow \infty} u_f(n, t) = 0, n \in \mathbb{Z}.$

Extension problem for $\log(-\Delta_d)$

Theorem (Betancor, DL-C, Rodríguez-Mesa, 2026)

Let $\{f(m)\}_{m \in \mathbb{Z}}$ s.t. $\sum_{m \in \mathbb{Z}} \frac{|f(m)|}{\sqrt{1+|m|}} < \infty$. We define

$$u_f(n, t) = \frac{1}{2} \int_0^\infty e^{u\Delta_d} f(x) \frac{e^{-\frac{t^2}{4u}}}{u} du, \quad n \in \mathbb{Z} \text{ and } t > 0.$$

Then,

- (i) $(\partial_t^2 + \frac{1}{t}\partial_t + \Delta_d)u_f(n, t) = 0, n \in \mathbb{Z} \text{ and } t > 0.$
- (ii) $-\lim_{t \rightarrow 0^+} t\partial_t u_f(n, t) = f(n), n \in \mathbb{Z}.$
- (iii) $\lim_{t \rightarrow \infty} u_f(n, t) = 0, n \in \mathbb{Z}.$
- (iv) $(\log(-\Delta_d)f)(n) = -2 \lim_{t \rightarrow 0^+} (u_f(n, t) + f(n) \log t) + f(n)K, n \in \mathbb{Z},$
where $K = 2(\log 2 - \gamma).$

Extension problem for $\log(-\Delta_d)$

Theorem (Betancor, DL-C, Rodríguez-Mesa, 2026)

Let $\{f(m)\}_{m \in \mathbb{Z}}$ s.t. $\sum_{m \in \mathbb{Z}} \frac{|f(m)|}{\sqrt{1+|m|}} < \infty$. We define

$$u_f(n, t) = \frac{1}{2} \int_0^\infty e^{u\Delta_d} f(x) \frac{e^{-\frac{t^2}{4u}}}{u} du, \quad n \in \mathbb{Z} \text{ and } t > 0.$$

Then,

- (i) $(\partial_t^2 + \frac{1}{t}\partial_t + \Delta_d)u_f(n, t) = 0, n \in \mathbb{Z} \text{ and } t > 0.$
- (ii) $-\lim_{t \rightarrow 0^+} t\partial_t u_f(n, t) = f(n), n \in \mathbb{Z}.$
- (iii) $\lim_{t \rightarrow \infty} u_f(n, t) = 0, n \in \mathbb{Z}.$
- (iv) $(\log(-\Delta_d)f)(n) = -2 \lim_{t \rightarrow 0^+} (u_f(n, t) + f(n) \log t) + f(n)K, n \in \mathbb{Z},$
where $K = 2(\log 2 - \gamma).$
- (v) $\lim_{t \rightarrow 0^+} \frac{u_f(n, t)}{\log t} = -f(n), n \in \mathbb{Z}.$

Some open problems:

- Which properties can we deduce for $\log(-\Delta_d)$ by using this extension theorem?

Some open problems:

- Which properties can we deduce for $\log(-\Delta_d)$ by using this extension theorem?
- Inspired by the extension theorem for $\mathcal{L}^s$ we can formulate an extension theorem for $\log(\mathcal{L})$:

$$\mathcal{L}^s \longrightarrow \log(\mathcal{L}) = \left. \frac{d}{ds} \mathcal{L}^s \right|_{s=0}$$

Some open problems:

- Which properties can we deduce for $\log(-\Delta_d)$ by using this extension theorem?
- Inspired by the extension theorem for $\mathcal{L}^s$ we can formulate an extension theorem for $\log(\mathcal{L})$:

$$\mathcal{L}^s \longrightarrow \log(\mathcal{L}) = \left. \frac{d}{ds} \mathcal{L}^s \right|_{s=0}$$

What if we start by the extension of a family $\{\phi_s(\mathcal{L})\}_s$, where each ϕ_s is a complete Bernstein function?

$$\phi_s(\mathcal{L}) \longrightarrow ?$$

References I

- [1] ARENDT, W., TER ELST, A. F. M., AND WARMA, M.
 Fractional powers of sectorial operators via the Dirichlet-to-Neumann operator.
Comm. Partial Differential Equations 43, 1 (2018), 1–24.
- [2] BANICA, V., GONZÁLEZ, M. D. M., AND SÁEZ, M.
 Some constructions for the fractional Laplacian on noncompact manifolds.
Rev. Mat. Iberoam. 31, 2 (2015), 681–712.
- [3] BHOWMIK, M., AND PUSTI, S.
 An extension problem and Hardy's inequality for the fractional Laplace-Beltrami operator on Riemannian symmetric spaces of noncompact type.
J. Funct. Anal. 282, 9 (2022), Paper No. 109413, 40.

References II

- [4] CAFFARELLI, L., AND SILVESTRE, L.
An extension problem related to the fractional Laplacian.
Comm. Partial Differential Equations 32, 7-9 (2007), 1245–1260.
- [5] CHANG, X., WANG, R., AND YAN, D.
Ground states for logarithmic Schrödinger equations on locally finite graphs.
J. Geom. Anal. 33, 7 (2023), Paper No. 211, 26.
- [6] CHEN, H.
On m -order logarithmic Laplacians and the applications.
Anal. Appl. (Singap.) 24, 2 (2026), 419–461.

References III

- [7] CHEN, H., CHEN, R., AND HUA, B.
Existence and convergence of least-energy solutions involving the logarithmic schrödinger operator.
Journal of Differential Equations 463 (2026), 114171.
- [8] CHEN, H., HAUER, H., AND WETH, T.
An extension problem for the logarithm Laplacian.
Preprint 2023 (arXiv:2312.15689).
- [9] CHEN, H., AND VÉRON, L.
Bounds for eigenvalues of the Dirichlet problem for the logarithmic Laplacian.
Adv. Calc. Var. 16, 3 (2023), 541–558.

References IV

[10] CHEN, H., AND VÉRON, L.

The Cauchy problem associated to the logarithmic Laplacian with an application to the fundamental solution.

J. Funct. Anal. 287, 3 (2024), Paper No. 110470, 72.

[11] CHEN, H., AND WETH, T.

The Dirichlet problem for the logarithmic Laplacian.

Comm. Partial Differential Equations 44, 11 (2019), 1100–1139.

[12] CHEN, R.

Logarithmic Laplacian on general Riemannian manifolds.

Preprint 2025 (arXiv:2506.19311).

[13] CHEN, R., AND XU, W.

The logarithm Laplacian on general graphs.

Preprint 2025 (arXiv:2507.05936).

References V

[14] GALÉ, J. E., MIANA, P. J., AND STINGA, P. R.

Extension problem and fractional operators: semigroups and wave equations.

J. Evol. Equ. 13, 2 (2013), 343–368.

[15] KWAŚNICKI, M., AND MUCHA, J.

Extension technique for complete Bernstein functions of the Laplace operator.

J. Evol. Equ. 18, 3 (2018), 1341–1379.

[16] STINGA, P. R., AND TORREA, J. L.

Extension problem and Harnack's inequality for some fractional operators.

Comm. Partial Differential Equations 35, 11 (2010), 2092–2122.

Thank you for your attention!

Operators in $L(c_0, \ell_1)$ and the BPBp: The long road towards the $L(c_0, \ell_1)$ case

José Luis Dávila



EARCO 2026

Encuentros de Análisis Real y Complejo



Instituto Universitario de Investigación
**de Matemáticas
y Aplicaciones**

Universidad Zaragoza

PID2022-137294NB-I00: Métodos Analíticos y Geométricos en Teoría de Operadores

May 29, 2026

Notation

- $L(X, Y) = \{T \in Y^X : T \text{ is linear and bounded}\}$

$$\|T\| = \sup\{\|T(x)\| : x \in S_X\}$$

- $X^* = L(X, \mathbb{K})$
- ℓ_∞^n denotes the space $\mathbb{K}^n$ endowed with the norm

$$\|x\|_\infty = \max\{|x(i)| : i \leq n\}$$

- c_0 denotes the space of sequences converging to zero in $\mathbb{K}^{\mathbb{N}}$ endowed with the norm

$$\|x\|_\infty = \sup\{|x(i)| : i \in \mathbb{N}\}.$$

Definition

An operator T in $L(X, Y)$ is said to attain its norm if there exists $x_0 \in S_X$ such that

$$\|T(x_0)\| = \|T\|.$$

We denote by $NA(X, Y)$ the collection of all norm attaining operators in $L(X, Y)$.

Example 1

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined as

$$T(x, y) = x + y \quad \forall (x, y) \in \mathbb{R}^2.$$

$$\|T\| = |T(1, 1)| = 2 \quad \text{if } T : (\mathbb{R}^2, \|\cdot\|_\infty) \rightarrow (\mathbb{R}, |\cdot|)$$

$$\|T\| = \left| T\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right| = \sqrt{2} \quad \text{if } T : (\mathbb{R}^2, \|\cdot\|_2) \rightarrow (\mathbb{R}, |\cdot|)$$

Definition

An operator T in $L(X, Y)$ is said to attain its norm if there exists $x_0 \in S_X$ such that

$$\|T(x_0)\| = \|T\|.$$

We denote by $NA(X, Y)$ the collection of all norm attaining operators in $L(X, Y)$.

Example 1

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined as

$$T(x, y) = x + y \quad \forall (x, y) \in \mathbb{R}^2.$$

$$\|T\| = |T(1, 1)| = 2 \quad \text{if } T : (\mathbb{R}^2, \|\cdot\|_\infty) \rightarrow (\mathbb{R}, |\cdot|)$$

$$\|T\| = \left| T\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right| = \sqrt{2} \quad \text{if } T : (\mathbb{R}^2, \|\cdot\|_2) \rightarrow (\mathbb{R}, |\cdot|)$$

Definition

An operator T in $L(X, Y)$ is said to attain its norm if there exists $x_0 \in S_X$ such that

$$\|T(x_0)\| = \|T\|.$$

We denote by $NA(X, Y)$ the collection of all norm attaining operators in $L(X, Y)$.

Example 2

Let $T : c_0 \rightarrow \mathbb{R}$ be defined as

$$T(x) = \sum_{n=1}^{\infty} \frac{x(n)}{2^n} \quad \forall x \in c_0.$$

$$\begin{aligned} \text{If } x \in S_{c_0}, \quad |T(x)| &= \left| \sum_{n=1}^{\infty} \frac{x(n)}{2^n} \right| \leq \sum_{n=1}^{\infty} \frac{|x(n)|}{2^n} < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 \\ \|T\| &= 1 \quad \text{and} \quad |T(x)| < 1 \end{aligned}$$

Definition

An operator T in $L(X, Y)$ is said to attain its norm if there exists $x_0 \in S_X$ such that

$$\|T(x_0)\| = \|T\|.$$

We denote by $NA(X, Y)$ the collection of all norm attaining operators in $L(X, Y)$.

Example 2

Let $T : c_0 \rightarrow \mathbb{R}$ be defined as

$$T(x) = \sum_{n=1}^{\infty} \frac{x(n)}{2^n} \quad \forall x \in c_0.$$

$$\text{If } x \in S_{c_0}, \quad |T(x)| = \left| \sum_{n=1}^{\infty} \frac{x(n)}{2^n} \right| \leq \sum_{n=1}^{\infty} \frac{|x(n)|}{2^n} < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$
$$\|T\| = 1 \quad \text{and} \quad |T(x)| < 1$$

Problem 1

Which Banach spaces X satisfy $NA(X, \mathbb{K}) = L(X, \mathbb{K})$?

Problem 2

Which Banach spaces X satisfy $\overline{NA(X, \mathbb{K})} = L(X, \mathbb{K})$?

Problem 1

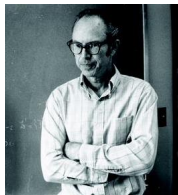
Which Banach spaces X satisfy $NA(X, \mathbb{K}) = L(X, \mathbb{K})$?

Problem 2

Which Banach spaces X satisfy $\overline{NA(X, \mathbb{K})} = L(X, \mathbb{K})$?



Errett Bishop (1928-1983)



Robert Phelps (1926-2013)

A PROOF THAT EVERY BANACH SPACE IS SUBREFLEXIVE

BY ERRETT BISHOP AND R. R. PHELPS

Communicated by Mahlon M. Day, August 19, 1960

Bishop-Phelps Theorem (1961)

If X is a Banach space, then

$$\overline{NA(X, \mathbb{K})} = L(X, \mathbb{K})$$

Problem 3

Which Banach spaces X and Y satisfy $\overline{NA(X, Y)} = L(X, Y)$?



Joram Lindenstrauss
(1936 - 2012)

ON OPERATORS WHICH ATTAIN THEIR NORM

BY
JORAM LINDENSTRAUSS⁽¹⁾

Lindenstrauss (1963)

If X is reflexive $\Rightarrow \overline{NA(X, Y)} = L(X, Y) \forall Y$ Banach.



Béla Bollobás
(1936 -)

AN EXTENSION TO THE THEOREM OF BISHOP AND PHELPS

BÉLA BOLLOBÁS

THEOREM 1. *Suppose $x \in S$, $f \in S'$ and $|f(x) - 1| \leq \varepsilon^2/2$ ($0 < \varepsilon < \frac{1}{2}$). Then there exist $y \in S$ and $g \in S'$ such that $g(y) = 1$, $\|f - g\| \leq \varepsilon$ and $\|x - y\| < \varepsilon + \varepsilon^2$.*

The Bishop-Phelps-Bollobás Theorem (1970)

If X is a Banach space then, for each $0 < \varepsilon < 1$ there exists $\eta = \eta(\varepsilon) > 0$ such that

$$\begin{aligned} \text{If } x_0 \in S_X, \quad f \in S_{\mathcal{L}(X, \mathbb{K})}, \quad |f(x_0) - 1| < \eta \quad \Rightarrow \\ \exists y_0 \in S_X, \quad g \in S_{\mathcal{L}(X, \mathbb{K})} : \\ g(y_0) = 1, \quad \|y_0 - x_0\| < \varepsilon \quad \text{and} \quad \|g - f\| < \varepsilon. \end{aligned}$$



The Bishop–Phelps–Bollobás theorem for operators

María D. Acosta^{a,1}, Richard M. Aron^{b,*,2}, Domingo García^{c,2},
Manuel Maestre^{c,2}

Definition (Acosta, Aron, Garcia and Maestre 2008)

(X, Y) has the **Bishop-Phelps-Bollobás property (BPBp)** for operators, if

for each $0 < \varepsilon < 1$, there exists $\eta = \eta(\varepsilon) > 0$ such that

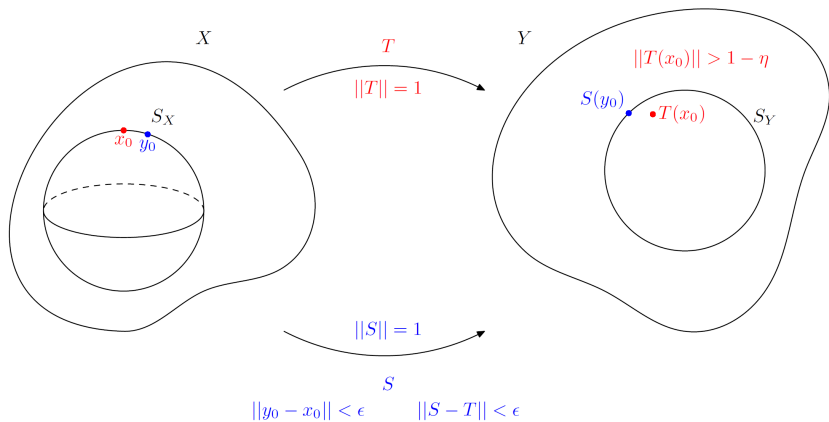
If $x_0 \in S_X$, $T \in S_{\mathcal{L}(X,Y)}$, $\|T(x_0)\| > 1 - \eta \Rightarrow$

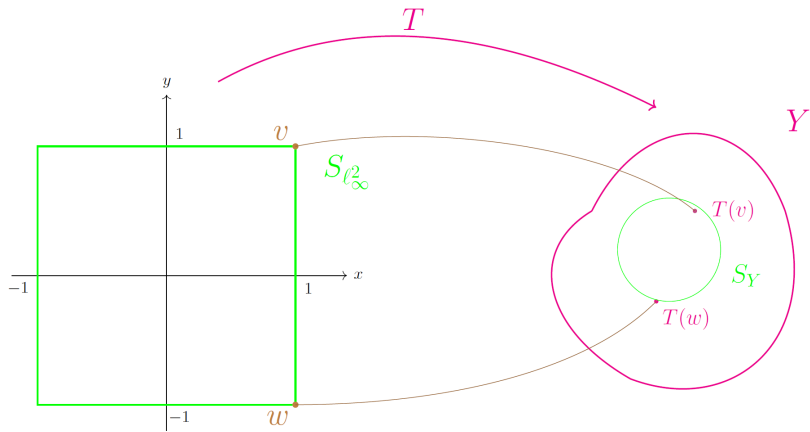
$\exists y_0 \in S_X$, $S \in S_{\mathcal{L}(X,Y)} :$

$\|S(y_0)\| = 1$, $\|y_0 - x_0\| < \varepsilon$ and $\|S - T\| < \varepsilon$.

Question

Which pairs (X, Y) satisfy the BPBp?





2022 The Bishop-Phelps-Bollobás theorem: an overview (Dantas, Sheldon - Maestre, Manuel - Roldán, Óscar)

BPBp	$\forall \mathcal{Y}$	$\mathbb{K}$	ℓ_1^m	ℓ_q^m	ℓ_∞^m	F.D.	c_0	ℓ_1	ℓ_q	ℓ_∞
$\forall \mathcal{X}$	✗	✓	✓		✓		✓	✗	✗	✓
$\mathbb{K}$	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
ℓ_1^2	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓
ℓ_1^n	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓
ℓ_p^n	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
ℓ_∞^n	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓
F.D.	✗	✓	✓	✓	✓	✓	✓			✓
c_0	✗	✓	✓	✓	✓		✓	$\checkmark_{\mathbb{C}}$	✓	✓
ℓ_1	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓
ℓ_p	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
ℓ_∞	✗	✓	✓	✓	✓		✓	$\checkmark_{\mathbb{C}}$	✓	✓
$L_1(\mu)$	✗	✓	✓	✓	✓	$\checkmark_\sigma$	✓	✓	✓	✓
$L_p(\mu)$	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
$L_\infty(\mu)$	✗	✓	✓	$\checkmark_{\mathbb{R} \mathbb{C}}$	✓		✓	$\checkmark_{\mathbb{C}}$	$\checkmark_{\mathbb{R} \mathbb{C}}$	✓
$C(K_1)$	✗	✓	✓	$\checkmark_{\mathbb{R} \mathbb{C}}$	✓		✓	$\checkmark_{\mathbb{C}}$	$\checkmark_{\mathbb{R} \mathbb{C}}$	✓
$C_0(L_1)$	✗	✓	✓	$\checkmark_{\mathbb{C}}$	✓		✓	$\checkmark_{\mathbb{C}}$	$\checkmark_{\mathbb{C}}$	✓

2022 The Bishop-Phelps-Bollobás theorem: an overview (Dantas, Sheldon - Maestre, Manuel - Roldán, Óscar)

BPBp	$L_1(v)$	$L_q(v)$	$L_\infty(v)$	$C(K_2)$	$C_0(L_2)$
$\forall \mathcal{X}$	$\times$	$\times$		$\times$	$\times$
$\mathbb{K}$	✓	✓	✓	✓	✓
ℓ_1^2	✓	✓	✓	✓	✓
ℓ_1^n	✓	✓	✓	✓	✓
ℓ_p^n	✓	✓	✓	✓	✓
ℓ_∞^n	✓ $\mathbb{C} \mid \mathbb{R}, 2_+$	✓	✓	✓	✓
F.D.			✓	✓	✓
c_0	✓ $\mathbb{C}$	✓	✓	✓	✓
ℓ_1	✓	✓	✓	✓	✓
ℓ_p	✓	✓	✓	✓	✓
ℓ_∞	✓ $\mathbb{C}$	✓	✓ $\mathbb{R}$	✓ $\mathbb{R}$	
$L_1(\mu)$	✓	✓	✓ 2_0		
$L_p(\mu)$	✓	✓	✓	✓	✓
$L_\infty(\mu)$	✓ $\mathbb{C}$	✓ $\mathbb{R} \mid \mathbb{C}$	✓ $\mathbb{R}$	✓ $\mathbb{R}$	
$C(K_1)$	✓ $\mathbb{C}$	✓ $\mathbb{R} \mid \mathbb{C}$	✓ $\mathbb{R}$	✓ $\mathbb{R}$	✓ $\mathbb{R}, 1_m$
$C_0(L_1)$	✓ $\mathbb{C}$	✓ $\mathbb{C}$	✓ $\mathbb{R}, 1_m$	✓ $\mathbb{R}, 1_m$	✓ $\mathbb{R}, 1_m$

A SIMPLE PROBLEM

(1) $|\text{Ext}(B_{\ell_\infty^3})| = 8$

$$-1 \leq -1/2 \leq 1/4 \leq 1/2 \leq 1$$

(2) $(1/2, -1/2, 1/4) \in B_{\ell_\infty^3}$

(3) Write the point
 $(1/2, -1/2, 1/4)$ as a convex
 combination of 4 points in $\text{Ext}(B_{\ell_\infty^3})$

$$\begin{aligned} (-1/2, 1/4, 1/2) &= \frac{-1/2 + 1}{2} v_1 \\ &\quad + \frac{1/4 + 1/2}{2} v_2 \\ &\quad + \frac{1/2 - 1/4}{2} v_3 \\ &\quad + \frac{1 - 1/2}{2} v_4 \end{aligned}$$

$$\begin{cases} v_1 = (1, 1, 1) \\ v_2 = (-1, 1, 1) \\ v_3 = (-1, -1, 1) \\ v_4 = (-1, -1, -1) \end{cases}$$

$$v_1 - v_2 + v_3 = (1, -1, 1)$$

$$v_1 - v_2 + v_4 = (1, -1, -1)$$

$$v_1 - v_3 + v_4 = (1, 1, -1)$$

$$v_2 - v_3 + v_4 = (-1, 1, -1)$$

A SIMPLE PROBLEM

(1) $|\text{Ext}(B_{\ell_\infty^3})| = 8$

(2) $(1/2, -1/2, 1/4) \in B_{\ell_\infty^3}$

(3) Write the point
 $(1/2, -1/2, 1/4)$ as a convex
 combination of 4 points in $\text{Ext}(B_{\ell_\infty^3})$

$$\begin{cases} v_1 = (1, 1, 1) \\ v_2 = (-1, 1, 1) \\ v_3 = (-1, -1, 1) \\ v_4 = (-1, -1, -1) \end{cases}$$

$$-1 \leq -1/2 \leq 1/4 \leq 1/2 \leq 1$$

$$\begin{aligned} (-1/2, 1/4, 1/2) &= \frac{-1/2 + 1}{2} v_1 \\ &\quad + \frac{1/4 + 1/2}{2} v_2 \\ &\quad + \frac{1/2 - 1/4}{2} v_3 \\ &\quad + \frac{1 - 1/2}{2} v_4 \end{aligned}$$

$$v_1 - v_2 + v_3 = (1, -1, 1)$$

$$v_1 - v_2 + v_4 = (1, -1, -1)$$

$$v_1 - v_3 + v_4 = (1, 1, -1)$$

$$v_2 - v_3 + v_4 = (-1, 1, -1)$$

A SIMPLE PROBLEM

(1) $|\text{Ext}(B_{\ell_\infty^3})| = 8$

$$-1 \leq -1/2 \leq 1/4 \leq 1/2 \leq 1$$

(2) $(1/2, -1/2, 1/4) \in B_{\ell_\infty^3}$

(3) Write the point
 $(1/2, -1/2, 1/4)$ as a convex
 combination of 4 points in $\text{Ext}(B_{\ell_\infty^3})$

$$\begin{aligned} (-1/2, 1/4, 1/2) &= \frac{-1/2 + 1}{2} v_1 \\ &+ \frac{1/4 + 1/2}{2} v_2 \\ &+ \frac{1/2 - 1/4}{2} v_3 \\ &+ \frac{1 - 1/2}{2} v_4 \end{aligned}$$

$$\begin{cases} v_1 = (1, 1, 1) \\ v_2 = (-1, 1, 1) \\ v_3 = (-1, -1, 1) \\ v_4 = (-1, -1, -1) \end{cases}$$

$$v_1 - v_2 + v_3 = (1, -1, 1)$$

$$v_1 - v_2 + v_4 = (1, -1, -1)$$

$$v_1 - v_3 + v_4 = (1, 1, -1)$$

$$v_2 - v_3 + v_4 = (-1, 1, -1)$$

A SIMPLE PROBLEM

(1) $|\text{Ext}(B_{\ell_\infty^3})| = 8$

$$-1 \leq -1/2 \leq 1/4 \leq 1/2 \leq 1$$

(2) $(1/2, -1/2, 1/4) \in B_{\ell_\infty^3}$

(3) Write the point
 $(1/2, -1/2, 1/4)$ as a convex
 combination of 4 points in $\text{Ext}(B_{\ell_\infty^3})$

$$\begin{aligned} (-1/2, 1/4, 1/2) &= \frac{-1/2 + 1}{2} v_1 \\ &\quad + \frac{1/4 + 1/2}{2} v_2 \\ &\quad + \frac{1/2 - 1/4}{2} v_3 \\ &\quad + \frac{1 - 1/2}{2} v_4 \end{aligned}$$

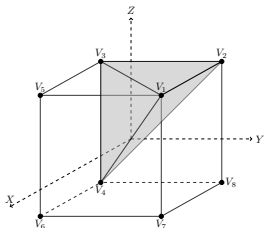
$$\begin{cases} v_1 = (1, 1, 1) \\ v_2 = (-1, 1, 1) \\ v_3 = (-1, -1, 1) \\ v_4 = (-1, -1, -1) \end{cases}$$

$$v_1 - v_2 + v_3 = (1, -1, 1)$$

$$v_1 - v_2 + v_4 = (1, -1, -1)$$

$$v_1 - v_3 + v_4 = (1, 1, -1)$$

$$v_2 - v_3 + v_4 = (-1, 1, -1)$$



If $P \in B_{\ell_\infty^3}$, then there exists an isometry I on ℓ_∞^3 such that

$$I(P) \in \text{co}(v_1, v_2, v_3, v_4)$$

Notation: The Set M_Y^n (Acosta, Dávila 2020)

If Y is a Banach space and n is a positive integer, we define

$$M_Y^n := \left\{ (y_i)_{i \leq n} \in Y^n : \sum_{j=1}^k (-1)^{j+1} y_{i_j} \in B_Y, \forall (i_1, \dots, i_k) \in \mathcal{I}_n \right\}.$$

AHSp- ℓ_∞^n

The space Y has the *approximate hyperplane sum property for ℓ_∞^n* if for every $0 < \varepsilon < 1$ there is $0 < \gamma_n(\varepsilon) < \varepsilon$ satisfying:

For every $(y_i)_{i \leq n} \in M_Y^n$, if $\exists$ a nonempty $A \subset \{1, \dots, n\}$ and $y^* \in S_{Y^*}$ such that $y^*(y_i) > 1 - \gamma_n(\varepsilon)$ for each $i \in A$, then $\exists (z_i)_{i \leq n} \in M_Y^n$ satisfying:

$$\|z_i - y_i\| < \varepsilon \quad \forall i \leq n \quad \text{and} \quad \left\| \sum_{i \in A} z_i \right\| = |A|.$$

Notation: The Set M_Y^n (Acosta, Dávila 2020)

If Y is a Banach space and n is a positive integer, we define

$$M_Y^n := \left\{ (y_i)_{i \leq n} \in Y^n : \sum_{j=1}^k (-1)^{j+1} y_{i_j} \in B_Y, \forall (i_1, \dots, i_k) \in \mathcal{I}_n \right\}.$$

AHSp- ℓ_∞^n

The space Y has the *approximate hyperplane sum property* for ℓ_∞^n if for every $0 < \varepsilon < 1$ there is $0 < \gamma_n(\varepsilon) < \varepsilon$ satisfying:

For every $(y_i)_{i \leq n} \in M_Y^n$, if $\exists$ a nonempty $A \subset \{1, \dots, n\}$ and $y^* \in S_{Y^*}$ such that $y^*(y_i) > 1 - \gamma_n(\varepsilon)$ for each $i \in A$, then $\exists (z_i)_{i \leq n} \in M_Y^n$ satisfying:

$$\|z_i - y_i\| < \varepsilon \quad \forall i \leq n \quad \text{and} \quad \left\| \sum_{i \in A} z_i \right\| = |A|.$$

Theorem (Acosta, Dávila 2020)

Let Y be a Banach space. The pair (ℓ_∞^n, Y) has the **BPBp** if and only if Y has the **AHSp**- ℓ_∞^n .

Spaces Y satisfying the AHSp- ℓ_∞^n

The *approximate hyperplane sum property* for ℓ_∞^n is satisfied by several important classes of Banach spaces Y :

- Finite-dimensional spaces
- Uniformly convex spaces
- $Y = L_1(\mu)$
- Spaces of continuous functions $C(K)$

Theorem (Acosta, Dávila 2020)

Let Y be a Banach space. The pair (ℓ_∞^n, Y) has the **BPBp** if and only if Y has the **AHSp**- ℓ_∞^n .

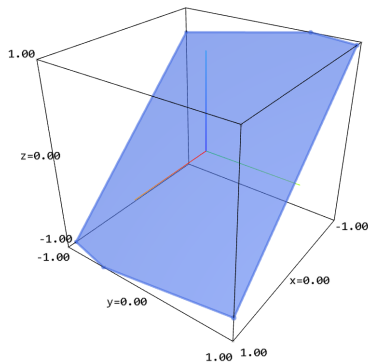
Spaces Y satisfying the AHSp- ℓ_∞^n

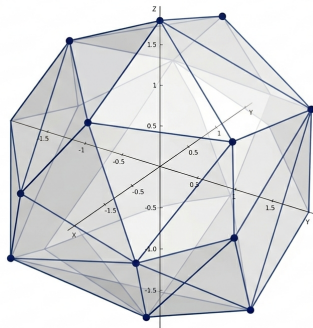
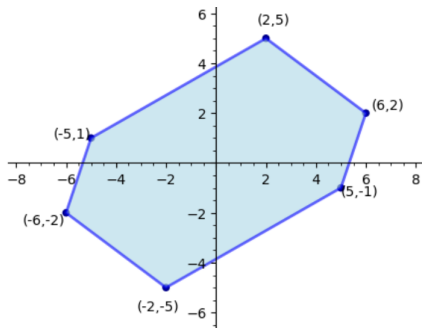
The *approximate hyperplane sum property* for ℓ_∞^n is satisfied by several important classes of Banach spaces Y :

- **Finite-dimensional spaces**
- **Uniformly convex spaces**
- $Y = L_1(\mu)$
- **Spaces of continuous functions** $C(K)$

Question: Inheritance of the BPBp to Subspaces

Let X and Y be Banach spaces. Suppose that the pair (X, Y) has the **BPBp**. If Z is a closed subspace of X , does the pair (Z, Y) necessarily inherit the **BPBp**?





Thank you for your attention

References



M.D. Acosta, R.M. Aron, D. García and M. Maestre (2008)

“The Bishop-Phelps-Bollobás theorem for operators”

J. Funct. Anal. 254, no. 11, 2780–2799.



M.D. Acosta, J. Becerra-Guerrero, D. García, S.K. Kim and M. Maestre (2015)

“The Bishop-Phelps-Bollobás property: a finite-dimensional approach”

Publ. Res. Inst. Math. Sci. 51, no. 1, 173–190.



M.D. Acosta and J.L. Dávila (2020)

“A basis of $\mathbb{R}^n$ with good isometric properties and some applications to denseness of norm attaining operators”

J. Funct. Anal. 279, 108602.



E. Bishop and R.R. Phelps (1961)

“A proof that every Banach space is subreflexive”

Bull. Amer. Math. Soc. 67, 97–98.



B. Bollobás (1970)

“An extension to the theorem of Bishop and Phelps”

Bull. Lond. Math. Soc. 2, 181–182.

References



J. Lindenstrauss (1963)

“On operators which attain their norm”

Israel J. Math. 1, 139–148.

λ -Forman-Ricci Curvature on graphs

Rosalio Reyes

join to J.A. Mendez-Bermudez, J.M. Rodriguez, J.M. Sigarreta, J.L. Sanchez-Santiesteban

May 29, 2026

Ricci Curvature

Fundamental concept in Riemannian geometry, characterizing local geometric properties by linking volume growth with geodesic dispersion.

Evolution of Discrete Curvatures

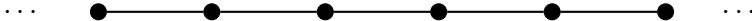
Several approaches exist to define curvature in graphs:

Ollivier-Ricci: Based on optimal transport and the Wasserstein metric.

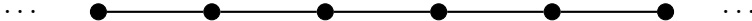
Forman-Ricci (F): Combinatorial approach based on the Bochner Laplacian.

$$F(uv) = 4 - \deg(u) - \deg(v)$$

Curvature 0



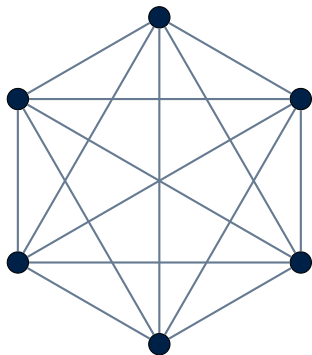
Curvature 0



$$F(uv) = 4 - \deg(u) - \deg(v) = 4 - 2 - 2 = 0.$$

Complete Graph K_6

A **complete graph** on n vertices (denoted as K_n) is a graph where every pair of distinct vertices is connected by a unique edge. For K_6 :



Geometric Properties of K_6

- **Vertices (n):** 6
- **Edges (m):** $\frac{6 \times 5}{2} = 15$
- **Degree ($\delta = \Delta$):** 5 (5-regular graph)

Curvature Connection

As every vertex has $\deg(v) = 5$, the standard Forman-Ricci curvature for any edge is highly negative:

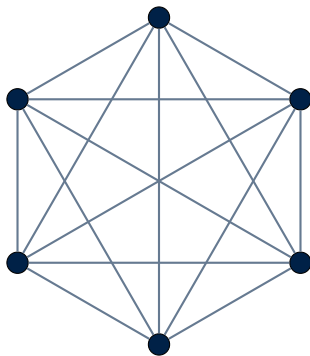
$$F(uv) = 4 - 5 - 5 = -6$$

Forman-Ricci (F): Combinatorial approach based on the Bochner Laplacian.

$$F(uv) = 4 - \deg(u) - \deg(v)$$

Augmented Forman-Ricci (F_3): Accounts for local cohesion via 3-cycles (triangles).

$$F_3(uv) = F(uv) + 3 \times (\# \text{ of triangles})$$



$$F_3(uv) = 4 - 5 - 5 + 3(4) = 6.$$

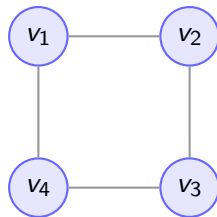
General Definition

$$F_\lambda(uv) = 4 - \deg(u) - \deg(v) + \lambda \cdot \#\{\triangle \text{ containing } uv\}$$

Versatility of the parameter:

- $\lambda = 0$: Original Forman-Ricci curvature.
- $\lambda = 3$: Augmented Forman-Ricci curvature.
- Allows tuning the importance of short cycles in the network.

Cycle C_4

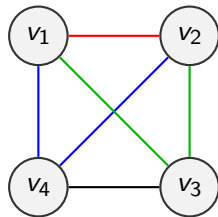


$$F(uv) = 4 - 2 - 2 = 0,$$

$$F_3(uv) = 4 - 2 - 2 = 0,$$

$$f_\lambda(uv) = 4 - 2 - 2 = 0.$$

Complete K_4



$$F(uv) = 4 - 3 - 3 = -2,$$

$$F_3(uv) = 4 - 3 - 3 + 3(2) = 0,$$

$$F_\lambda(uv) = 4 - 3 - 3 + \lambda(2) = -2 + 2\lambda.$$

Properties in Specific Graphs

The behavior of F_λ varies significantly depending on the topology:

- **Complete Graphs (K_n):** $F_\lambda = 6 - 2n + \lambda(n - 2)$.
- **Star Graphs (S_n):** $F_\lambda = 4 - n$ (Independent of λ , as there are no triangles).
- **Cycles ($C_n, n \geq 4$):** $F_\lambda = 0$.
- **Hypercubes (Q_n):** $F_\lambda = 4 - 2n$.

F_λ efficiently detects "density" and connectivity types.

Let G be a graph with minimum degree δ and maximum degree Δ . For any $\lambda \in \mathbb{R}$, the λ -Forman-Ricci curvature satisfies:

Proposition 2.12

- **Case 1:** $\lambda \geq 0$

$$F_{\lambda}(uv) \leq 4 - 2\delta + \lambda(\Delta - 1)$$

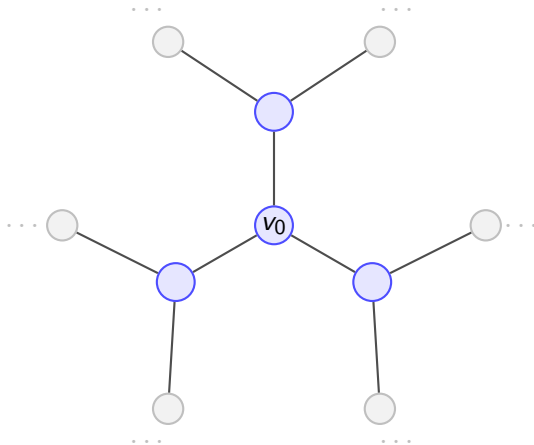
Equality is achieved if and only if G is a complete graph K_n .

- **Case 2:** $\lambda < 0$

$$F_{\lambda}(uv) \leq 4 - 2\delta$$

Equality is achieved if and only if G is a δ -regular tree.

3-regular tree



Proposition 2.1: Bounds on Maximum Degree

Let G be a graph such that $F_\lambda \geq k$ for some real constants λ, k , and let Δ denote the maximum degree of G .

Proposition

① If $\lambda \leq 0$, then:

$$\Delta \leq 3 - k$$

② If $0 \leq \lambda < 2$, then:

$$\Delta \leq \frac{2}{2 - \lambda}(4 - k) - 1$$

- This proposition establishes an upper bound on the maximum degree Δ of a network based on a lower bound of its λ -Forman-Ricci curvature.

Curvature Lower Bounds for Infinite vs Finite Graphs

Let G be a graph with at least 2 edges, minimum degree δ , maximum degree Δ , and $k \in \mathbb{R}$ such that $F_\lambda \geq k$ with $\lambda \geq 0$.

Theorem

- If $\Delta = \infty$, then:

$$\lambda \geq 1$$

- If $\Delta < \infty$, then:

$$\lambda \geq \frac{k - 4 + \delta + \Delta}{\Delta - 1}$$

- **Implication:** For networks with infinite maximum degree, the weighting parameter λ must be at least 1 to sustain a uniform lower bound on the curvature.

Optimal Lower Bounds in Terms of Vertices

Let G be a graph with n vertices and $\lambda \in \mathbb{R}$.

Theorem (Lower Bounds)

① If $\lambda < 1$, then:

$$F_\lambda \geq 6 - 2n + \lambda(n - 2)$$

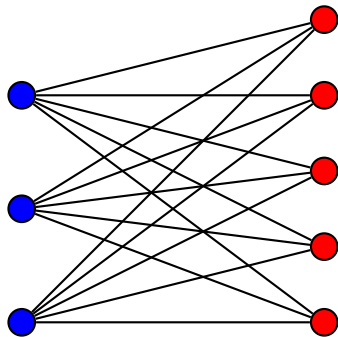
The equality holds if and only if G is the complete graph K_n .

② If $\lambda \geq 1$, then:

$$F_\lambda \geq 4 - n$$

If $\lambda > 1$, the equality holds if and only if G is a complete bipartite graph $K_{t,n-t}$ for some $1 \leq t \leq n$.

Bipartite complete graph $K_{3,5}$



Optimal Upper Bounds in Terms of Vertices

Let G be a graph with n vertices and $\lambda \in \mathbb{R}$.

Theorem (Upper Bounds)

- 1 If $\lambda \leq 1$, then $F_\lambda \leq 1$ unless $G = P_2$.
 - If $\lambda < 1$, equality holds $\iff G = P_3$.
 - If $\lambda = 1$, equality holds $\iff G$ is either P_3 or C_3 .
- 2 If $1 < \lambda < 2$, then $F_\lambda \leq \lambda$ unless $G = P_2$. Equality holds $\iff G = C_3$.

Metric graph

The *distance* $d_G(x, y)$ between two vertices x and y , in a connected graph G , is the minimum length of the paths joining x and y in G .

The *diameter* of G , denoted by $diam(G)$, is the maximum distance among all pairs of vertices of G .

Discrete Bonnet-Myers Theorem

In Riemannian geometry, positive Ricci curvature implies a compact manifold with a bounded diameter. The following theorem presents the discrete analogue for graphs:

Diameter Bound

Let G be a connected graph. Assume that $F_\lambda(uv) \geq k$ for all $uv \in E(G)$ under one of the following conditions:

- 1 $\lambda \leq 0$ and $k > 0$
- 2 $0 \leq \lambda < 2$ and $k > \lambda$

Then, the diameter of the graph is bounded by:

$$\text{diam}(G) \leq 2$$

Polynomial volume growth.

Given any graph G , $u \in V(G)$ and $\rho > 0$, define

$$B_G(u, \rho) = \{v \in V(G) : d_G(v, u) < \rho\}, \quad \text{vol}(B_G(u, \rho)) = \# B_G(u, \rho).$$

Note that $\text{vol}(B_G(u, \rho)) = 1$ for every $0 < \rho \leq 1$, and $\text{vol}(B_G(u, \rho)) = 1 + \deg(u)$ for every $1 < \rho \leq 2$.

The *polynomial growth order* of a graph G as

$$\inf \left\{ k \geq 0 : \limsup_{\rho \rightarrow \infty} \rho^{-k} \text{vol}(B_G(u, \rho)) < \infty \right\}.$$

One can check that this definition does not depend on the choice of $u \in V(G)$.

We say that G has (at most) *polynomial volume growth* if its polynomial growth order is finite.

Manifolds with non-negative Ricci curvature exhibit at most polynomial volume growth. Theorem 3.8 introduces this structural property into the discrete domain:

Theorem 3.8

Let G be a graph, and let $\lambda < 4/3$ such that $F_\lambda(uv) \geq 0$ for all edges $uv \in E(G)$.

Then, G is either a **finite graph** or its polynomial growth order is exactly **1**.

- This highlights a powerful constraint: non-negative λ -curvature prevents exponential explosion or high-dimensional polynomial expansion in infinite graph topologies.

Isoperimetric constant

If G is a (finite or infinite) graph, we define the *isoperimetric constant* of G by

$$i(G) := \min \frac{|\partial X|}{|X|},$$

where $|A|$ denotes the cardinality of the set A , ∂X denotes the set of edges in $E(G)$ having one end in X and the other end in $V(G) \setminus X$, and the minimum is taken over all non-empty finite subsets $X \subset V(G)$ satisfying $|X| \leq \lceil |V(G)|/2 \rceil$

Isoperimetric Constant and Curvature Bounds

The Cheeger-like isoperimetric constant $i(G)$ relates to network boundary sizes. For graphs with higher order, the parameter λ bounds the global geometric expansion:

Theorem

If G is a graph with more than 3 vertices and $0 \leq \lambda \leq 2$, then:

$$F_\lambda \leq 2 - (2 - \lambda)i(G)$$

- **Sharpness:** The equality in this inequality is strictly attained if G is the complete graph K_4 .
- This inequality establishes a direct link between graph spectral/isoperimetric properties ($i(G)$) and the local λ -Forman-Ricci curvature.

Consider $uv \in E(G)$ and $X = \{u, v\}$. Since $|V(G)| \geq 4$,

$$i(G) \leq \frac{|\partial X|}{|X|} = \frac{1}{2}(\deg(u) + \deg(v) - 2).$$

Consider $uv \in E(G)$ and $X = \{u, v\}$. Since $|V(G)| \geq 4$,

$$i(G) \leq \frac{|\partial X|}{|X|} = \frac{1}{2}(\deg(u) + \deg(v) - 2).$$

$$\#\{\text{triangles} \ni uv\} \leq \min\{\deg(u), \deg(v)\} - 1 \leq \frac{1}{2}(\deg(u) + \deg(v)) - 1,$$

and $\lambda \geq 0$, we have for any $uv \in E(G)$,

Consider $uv \in E(G)$ and $X = \{u, v\}$. Since $|V(G)| \geq 4$,

$$i(G) \leq \frac{|\partial X|}{|X|} = \frac{1}{2}(\deg(u) + \deg(v) - 2).$$

$$\#\{\text{triangles} \ni uv\} \leq \min\{\deg(u), \deg(v)\} - 1 \leq \frac{1}{2}(\deg(u) + \deg(v)) - 1,$$

and $\lambda \geq 0$, we have for any $uv \in E(G)$,

$$\begin{aligned} F_\lambda(uv) &= 4 - \deg(u) - \deg(v) + \lambda \cdot \#\{\text{triangles} \ni uv\} \\ &\leq 4 - \deg(u) - \deg(v) + \frac{\lambda}{2}(\deg(u) + \deg(v) - 2) \\ &= 2 - (2 - \lambda) \frac{1}{2}(\deg(u) + \deg(v) - 2). \end{aligned}$$

Thank You!

Any Questions?

On λ -Forman-Ricci curvature of networks.

J.A. Mendez-Bermudez R. Reyes J.M. Rodríguez J.M. Sigarreta
RAIRO Operations Research, 59 (2025)

Geometric properties in graphs: λ -Forman-Ricci curvature.

R. Reyes J.M. Rodríguez J.M. Sigarreta J.L. Sanchez-Santiesteban
submitted

A FRACTAL LOCAL SMOOTHING CONJECTURE

FOR THE WAVE EQUATION

XXIII EARCO

LOGROÑO, 29 MAYO 2026

DAVID BELTRAN
UNIVERSITAT DE VALÈNCIA.

Wave equation on $\mathbb{R}^d$

$$\begin{cases} \partial_{tt}^2 u = \Delta u \\ u(x, 0) = f_0(x), \quad \partial_t u(x, 0) = f_1(x) \end{cases}$$

$$f_0, f_1 : \mathbb{R}^d \rightarrow \mathbb{C}$$

By Fourier transform

$$u(x, t) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} \cos(t|\xi|) \hat{f}_0(\xi) d\xi + \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} \sin(t|\xi|) \frac{\hat{f}_1(\xi)}{|\xi|} d\xi$$

Wave equation on $\mathbb{R}^d$

$$\begin{cases} \partial_{tt}^2 u = \Delta u \\ u(x, 0) = f_0(x), \quad \partial_t u(x, 0) = f_1(x) \end{cases}$$

$$f_0, f_1 : \mathbb{R}^d \rightarrow \mathbb{C}$$

By Fourier transform

$$u(x, t) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} \cos(t|\xi|) \hat{f}_0(\xi) d\xi + \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} \sin(t|\xi|) \frac{\hat{f}_1(\xi)}{|\xi|} d\xi$$

$$e^{it\sqrt{-\Delta}} f(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi + it|\xi|} \hat{f}(\xi) d\xi$$

$$u(x, t) = e^{it\sqrt{-\Delta}} f_+(x) + e^{-it\sqrt{-\Delta}} f_-(x), \quad \text{where} \quad f_{\pm} = \frac{1}{2} (f_0 \mp i(\sqrt{-\Delta})^{-1} f_1)$$

Fixed time estimates

$$e^{it\sqrt{-\Delta}} f(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi + it|\xi|} \hat{f}(\xi) d\xi$$

Plancherel : $\| e^{it\sqrt{-\Delta}} f \|_{L^2(\mathbb{R}^d)} = \| f \|_{L^2(\mathbb{R}^d)}$

Fixed time estimates

$$e^{it\sqrt{-\Delta}} f(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi + it|\xi|} \hat{f}(\xi) d\xi$$

Plancherel : $\| e^{it\sqrt{-\Delta}} f \|_{L^2(\mathbb{R}^d)} = \| f \|_{L^2(\mathbb{R}^d)}$

Question : Are there any other size estimates ?

Fixed time estimates

$$e^{it\sqrt{-\Delta}} f(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi + it|\xi|} \hat{f}(\xi) d\xi$$

Plancherel : $\| e^{it\sqrt{-\Delta}} f \|_{L^2(\mathbb{R}^d)} = \| f \|_{L^2(\mathbb{R}^d)}$

Question : Are there any other size estimates ?

$$d=1$$

$$x \cdot \xi + t|\xi| = (x+t)\xi \mathbb{1}_{\xi \geq 0} + (x-t)\xi \mathbb{1}_{\xi < 0}$$

$$e^{it\sqrt{-\Delta}} f(x) = P_+ f(x+t) + P_- f(x-t)$$

"D'Alembert's formula"

$$\| e^{it\sqrt{-\Delta}} f \|_{L^p(\mathbb{R})} \lesssim \| f \|_{L^p(\mathbb{R})} \quad 1 < p < \infty$$

Fixed time estimates

$$e^{it\sqrt{-\Delta}} f(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi + it|\xi|} \hat{f}(\xi) d\xi$$

Plancherel : $\| e^{it\sqrt{-\Delta}} f \|_{L^2(\mathbb{R}^d)} = \| f \|_{L^2(\mathbb{R}^d)}$

Question : Are there any other size estimates ?

$d \geq 2$ Peral, Miyachi (1980) $1 < p < \infty$, $1 \leq t \leq 2$

$$\| e^{it\sqrt{-\Delta}} f \|_{L^p(\mathbb{R}^d)} \lesssim \| f \|_{L^p_{s_p}(\mathbb{R}^d)}$$

where $s_p = (d-1) \left| \frac{1}{p} - \frac{1}{2} \right|$

Fixed time estimates

$$e^{it\sqrt{-\Delta}} f(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi + it|\xi|} \hat{f}(\xi) d\xi$$

Plancherel : $\| e^{it\sqrt{-\Delta}} f \|_{L^2(\mathbb{R}^d)} = \| f \|_{L^2(\mathbb{R}^d)}$

Question : Are there any other size estimates ?

$d \geq 2$ Peral, Miyachi (1980) $1 < p < \infty$, $1 \leq t \leq 2$

$$\| e^{it\sqrt{-\Delta}} f_j \|_{L^p(\mathbb{R}^d)} \lesssim 2^{js_p} \| f_j \|_{L^p(\mathbb{R}^d)}$$

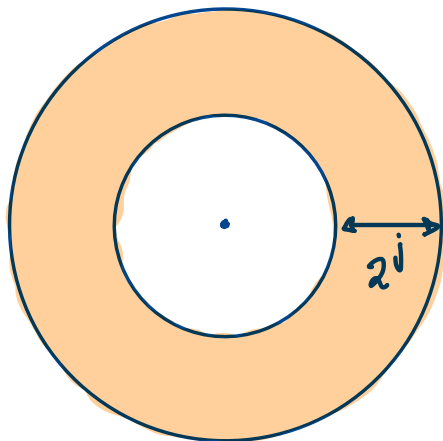
where $s_p = (d-1) \left| \frac{1}{p} - \frac{1}{2} \right|$, if $\text{supp}(\hat{f}_j) \subseteq \{ |\xi| \sim 2^j \}$, $j \geq 1$

Why $s_p = (d-1) \left| \frac{1}{p} - \frac{1}{2} \right|$?

$$\| e^{it\sqrt{-\Delta}} f_j \|_{L^p(\mathbb{R}^d)} \lesssim 2^{js_p} \| f \|_{L^p(\mathbb{R}^d)}$$

$$e^{it\sqrt{-\Delta}} f_j = K_j^t * f_j, \text{ where } K_j(x) = \int_{\mathbb{R}^d} e^{i(x \cdot \xi + t|\xi|)} \psi(2^{-j}\xi) d\xi$$

ξ -space



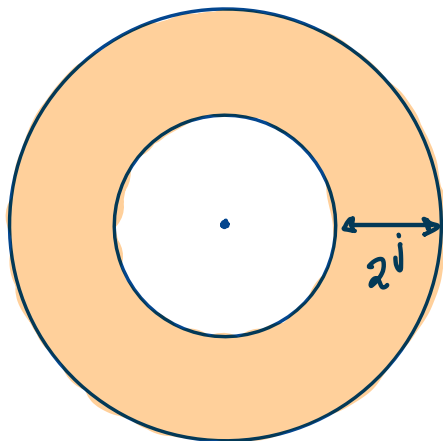
Why $s_p = (d-1) \left| \frac{1}{p} - \frac{1}{2} \right|$?

$$\| e^{it\sqrt{-\Delta}} f_j \|_{L^p(\mathbb{R}^d)} \lesssim 2^{js_p} \| f \|_{L^p(\mathbb{R}^d)}$$

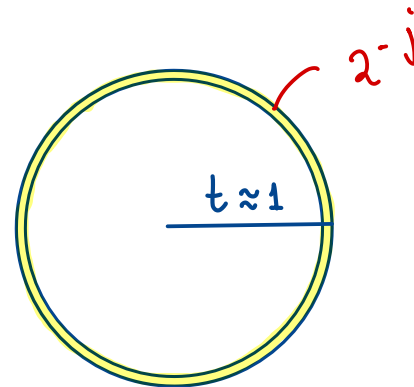
$$e^{it\sqrt{-\Delta}} f_j = K_j^t * f_j, \text{ where } K_j(x) = \int_{\mathbb{R}^d} e^{i(x \cdot \xi + t|\xi|)} \psi(2^{-j}\xi) d\xi$$

• K_j^t concentrates on $| |x| - t | \lesssim 2^{-j}$

ξ -space



x -space



Why $s_p = (d-1) \left| \frac{1}{p} - \frac{1}{2} \right|$?

$$\| e^{it\sqrt{-\Delta}} f_j \|_{L^p(\mathbb{R}^d)} \lesssim 2^{js_p} \| f \|_{L^p(\mathbb{R}^d)} \quad (\text{FT})$$

$$e^{it\sqrt{-\Delta}} f_j = K_j^t * f_j, \text{ where } K_j^t(x) = \int_{\mathbb{R}^d} e^{i(x \cdot \xi + t|\xi|)} \psi(2^{-j}\xi) d\xi$$

- K_j^t concentrates on $||x| - t| \lesssim 2^{-j}$
- $\| K_j^t \|_2 = \| \widehat{K_j^t} \|_2 = \| \psi(2^{-j} \cdot) \|_2 \approx 2^{jd/2}$

Why $s_p = (d-1) \left| \frac{1}{p} - \frac{1}{2} \right|$?

$$\| e^{it\sqrt{-\Delta}} f_j \|_{L^p(\mathbb{R}^d)} \lesssim 2^{js_p} \| f \|_{L^p(\mathbb{R}^d)} \quad (\text{FT})$$

$$e^{it\sqrt{-\Delta}} f_j = K_j^t * f_j, \text{ where } K_j^t(x) = \int_{\mathbb{R}^d} e^{i(x \cdot \xi + t|\xi|)} \psi(2^{-j}\xi) d\xi$$

- K_j^t concentrates on $||x| - t| \lesssim 2^{-j}$
- $\| K_j^t \|_2 = \| \widehat{K_j^t} \|_2 = \| \psi(2^{-j} \cdot) \|_2 \approx 2^{jd/2}$
- $\| K_j^t \|_1 \leq |\text{supp } K_j^t|^{1/2} \| K_j^t \|_2 \approx 2^{j(d-1)/2} \Rightarrow (\text{FT})$

Why $s_p = (d-1) \left| \frac{1}{p} - \frac{1}{2} \right|$?

$$\| e^{it\sqrt{-\Delta}} f_j \|_{L^p(\mathbb{R}^d)} \lesssim 2^{js_p} \| f \|_{L^p(\mathbb{R}^d)} \quad (\text{FT})$$

$$e^{it\sqrt{-\Delta}} f_j = K_j^t * f_j, \text{ where } K_j^t(x) = \int_{\mathbb{R}^d} e^{i(x \cdot \xi + t|\xi|)} \psi(2^{-j}\xi) d\xi$$

- K_j^t concentrates on $| |x| - t | \lesssim 2^{-j}$
- $\| K_j^t \|_2 = \| \widehat{K_j^t} \|_2 = \| \psi(2^{-j} \cdot) \|_2 \approx 2^{jd/2}$
- $\| K_j^t \|_1 \leq |\text{supp } K_j^t|^{1/2} \| K_j^t \|_2 \approx 2^{j(d-1)/2} \Rightarrow (\text{FT})$
- U. P. + radiality $\Rightarrow |K_j^t|$ essentially constant

Why $s_p = (d-1) \left| \frac{1}{p} - \frac{1}{2} \right|$?

$$\| e^{it\sqrt{-\Delta}} f_j \|_{L^p(\mathbb{R}^d)} \lesssim 2^{js_p} \| f \|_{L^p(\mathbb{R}^d)} \quad (\text{FT})$$

$$e^{it\sqrt{-\Delta}} f_j = K_j^t * f_j, \text{ where } K_j^t(x) = \int_{\mathbb{R}^d} e^{i(x \cdot \xi + t|\xi|)} \psi(2^{-j}\xi) d\xi$$

- K_j^t concentrates on $| |x| - t | \lesssim 2^{-j}$
- $\| K_j^t \|_2 = \| \widehat{K_j^t} \|_2 = \| \psi(2^{-j} \cdot) \|_2 \approx 2^{jd/2}$
- $\| K_j^t \|_1 \lesssim | \text{supp } K_j^t |^{1/2} \| K_j^t \|_2 \approx 2^{j(d-1)/2} \Rightarrow (\text{FT})$
- U. P. + radiality $\Rightarrow |K_j^t|$ essentially constant
- $\| K_j^t \|_p \approx | \text{supp } K_j^t |^{1/p} \| K_j^t \|_\infty$

Why $s_p = (d-1) \left| \frac{1}{p} - \frac{1}{2} \right|$?

$$\| e^{it\sqrt{-\Delta}} f_j \|_{L^p(\mathbb{R}^d)} \lesssim 2^{js_p} \| f \|_{L^p(\mathbb{R}^d)} \quad (\text{FT})$$

$$e^{it\sqrt{-\Delta}} f_j = K_j^t * f_j, \text{ where } K_j^t(x) = \int_{\mathbb{R}^d} e^{i(x \cdot \xi + t|\xi|)} \psi(2^{-j}\xi) d\xi$$

- K_j^t concentrates on $||x| - t| \lesssim 2^{-j}$
- $\| K_j^t \|_2 = \| \widehat{K_j^t} \|_2 = \| \psi(2^{-j} \cdot) \|_2 \approx 2^{jd/2}$
- $\| K_j^t \|_1 \leq |\text{supp } K_j^t|^{1/2} \| K_j^t \|_2 \approx 2^{j(d-1)/2} \Rightarrow (\text{FT})$
- U. P. + radiality $\Rightarrow |K_j^t|$ essentially constant
- $\| K_j^t \|_p \approx |\text{supp } K_j^t|^{1/p} \| K_j^t \|_\infty \approx |\text{supp } K_j^t|^{\frac{1}{p} - \frac{1}{2}} \| K_j^t \|_2 \approx 2^{j\frac{d}{2} + j(\frac{1}{2} - \frac{1}{p})}$

Why $s_p = (d-1) \left| \frac{1}{p} - \frac{1}{2} \right|$?

$$\| e^{it\sqrt{-\Delta}} f_j \|_{L^p(\mathbb{R}^d)} \lesssim 2^{js_p} \| f \|_{L^p(\mathbb{R}^d)} \quad (\text{FT})$$

Best possible : Take $\hat{g}_j(\xi) := e^{-it|\xi|} \psi(2^j \xi)$

$$\Rightarrow g_j = K_j^t \quad \text{and} \quad e^{it\sqrt{-\Delta}} g_j(x) = 2^{jd} \psi(2^j x)$$

Why $s_p = (d-1) \left| \frac{1}{p} - \frac{1}{2} \right|$?

$$\| e^{it\sqrt{-\Delta}} f_j \|_{L^p(\mathbb{R}^d)} \lesssim 2^{js_p} \| f \|_{L^p(\mathbb{R}^d)} \quad (\text{FT})$$

Best possible : Take $\hat{g}_j(\xi) := e^{-it|\xi|} \psi(2^j \xi)$

$$\Rightarrow g_j = K_j^t \quad \text{and} \quad e^{it\sqrt{-\Delta}} g_j(x) = 2^{jd} \check{\psi}(2^j x)$$

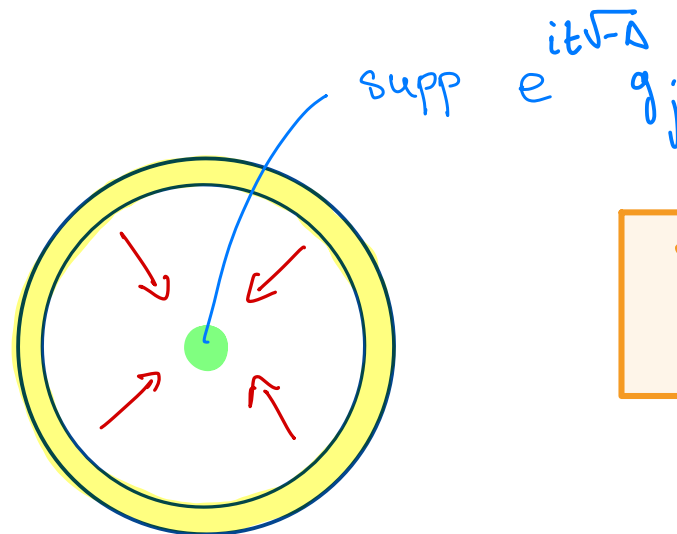
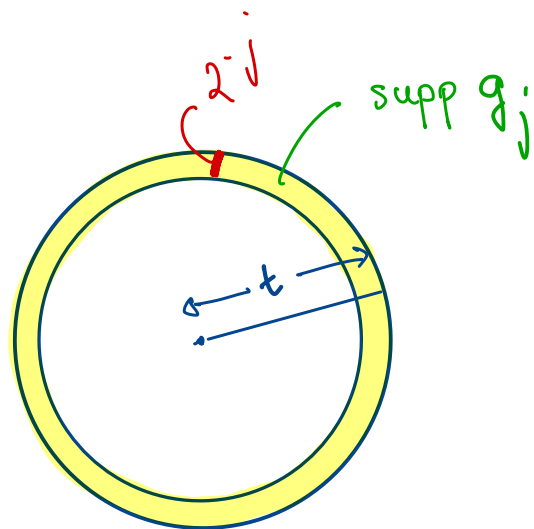
$$\text{Thus} \quad \frac{\| e^{it\sqrt{-\Delta}} g_j \|_p}{\| g_j \|_p} \approx \frac{2^{jd} \| \check{\psi}(2^j \cdot) \|_p}{\| K_j^t \|_p} \approx 2^{j(d-1)(\frac{1}{2}-\frac{1}{p})}$$

Why $s_p = (d-1) \left| \frac{1}{p} - \frac{1}{2} \right|$?

$$\| e^{it\sqrt{-\Delta}} f_j \|_{L^p(\mathbb{R}^d)} \lesssim 2^{js_p} \| f_j \|_{L^p(\mathbb{R}^d)} \quad (FT)$$

Best possible : Take $\hat{g}_j(\xi) := e^{-it|\xi|} \psi(2^j \xi)$

$$\Rightarrow g_j = K_j^t \quad \text{and} \quad e^{it\sqrt{-\Delta}} g_j(x) = 2^{jd} \psi(2^j x)$$



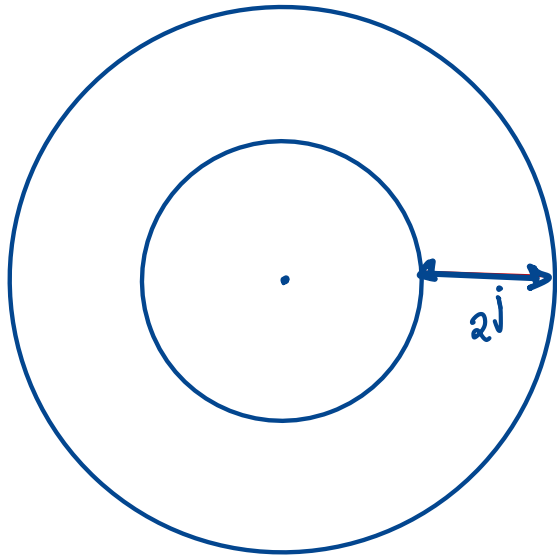
FOCUSING
EXAMPLE

Focusing example through wave packets

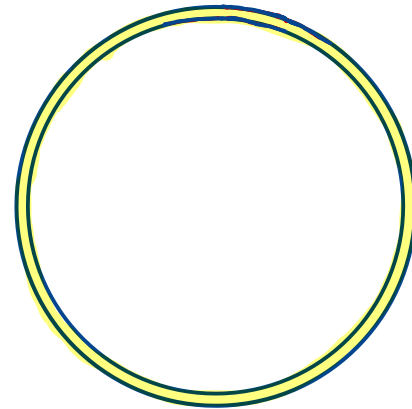
Example : $\hat{g}_j(\xi) = e^{-i|\xi|} \psi(2^j \xi)$

$$|\xi| = \xi_1 \sqrt{1 + \frac{\xi_2^2}{\xi_1^2}} \approx \xi_1 + \frac{\xi_2^2}{2\xi_1}$$

ξ - space



x - space

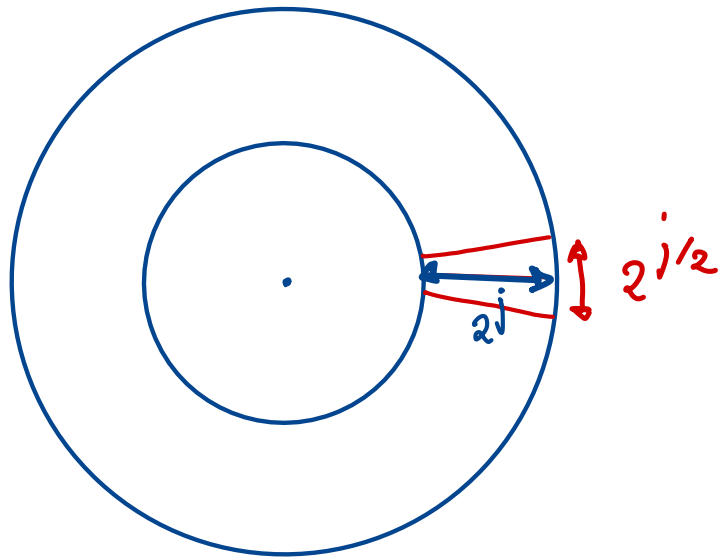


Focusing example through wave packets

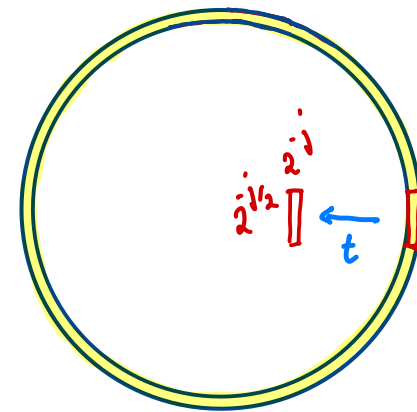
Example : $\hat{g}_j(\xi) = e^{-i|\xi|} \psi(2^j \xi)$

$$|\xi| = \xi_1 \sqrt{1 + \frac{\xi_2^2}{\xi_1^2}} \approx \xi_1 + \frac{\xi_2^2}{2\xi_1}$$

ξ - space



x - space

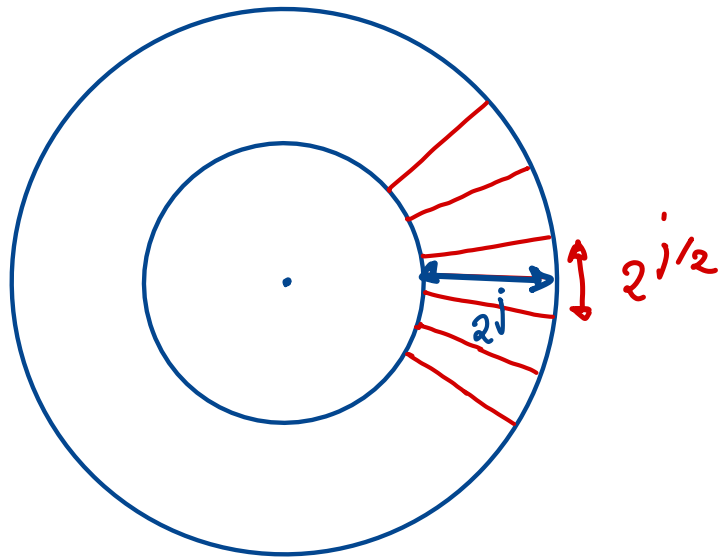


"d'Alembert's formula"

Focusing example through wave packets

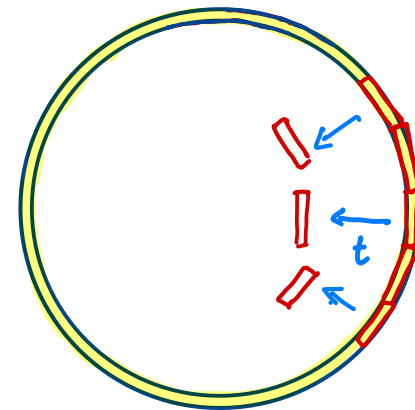
Example : $\hat{g}_j(\xi) = e^{-i|\xi|} \psi(2^j \xi)$

ξ - space



$$g_j = \sum_{\nu} g_j^{\nu}$$

x - space



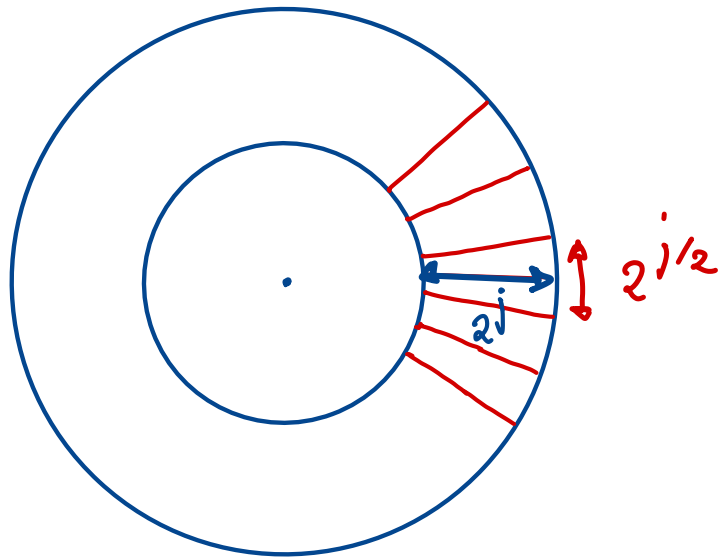
"d'Alembert's formula"

$e^{it\sqrt{-\Delta}} g_j^{\nu}$ is a translation by t
in the direction e_j^{ν}

Focusing example through wave packets

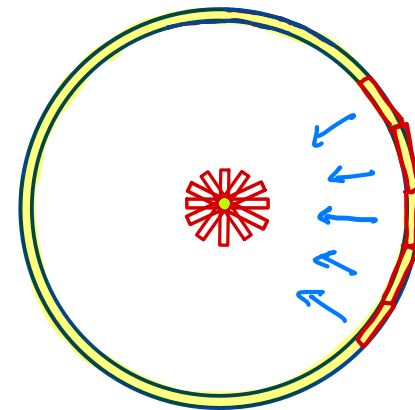
Example : $\hat{g}_j(\xi) = e^{-i|\xi|} \psi(2^j \xi)$

ξ - space



$$g_j = \sum_{\nu} g_j^{\nu}$$

x - space



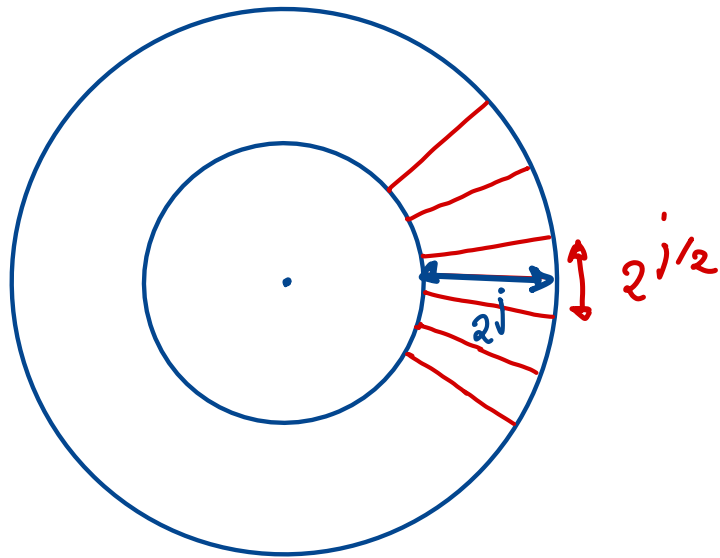
"d'Alembert's formula"

$e^{it\sqrt{-\Delta}} g_j^{\nu}$ is a translation by t
in the direction e_j^{ν}

Focusing example through wave packets

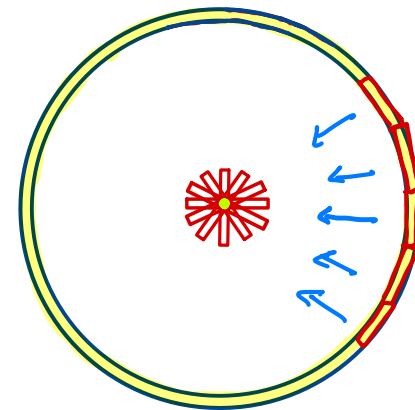
Example : $\hat{g}_j(\xi) = e^{-i|\xi|} \psi(2^{-j}\xi)$

ξ - space



$$g_j = \sum_{\nu} g_{j,\nu}$$

x - space



"d'Alembert's formula"

Highest overlap for $1 \leq t \leq 1 + 2^{-j}$
then fades away

Local smoothing estimates

Let $\{t_\mu\}$ a set of 2^{-j} -separated points in $[1, 2]$

Sogge's question : can we have

$$\left(\sum_{\mu} \| e^{it_{\mu}\sqrt{-\Delta}} f_j \|_{L^p(\mathbb{R}^d)}^p \right)^{1/p} \lesssim 2^{js_p} \| f \|_{L^p(\mathbb{R}^d)}$$

for some large p , where $s_p = (d-1) \left(\frac{1}{2} - \frac{1}{p} \right)$?

Local smoothing estimates

Let $\{t_\mu\}$ a set of 2^{-j} -separated points in $[1, 2]$

Sogge's question : can we have

$$\left(\sum_{\mu} \| e^{it_{\mu} \sqrt{-\Delta}} f_j \|_{L^p(\mathbb{R}^d)}^p \right)^{1/p} \lesssim 2^{js_p} \| f \|_{L^p(\mathbb{R}^d)}$$

for some large p , where $s_p = (d-1) \left(\frac{1}{2} - \frac{1}{p} \right)$?

- ① It is trivial to have the inequality with $2^{js_p + j/p}$.
- ② large p because it's when overlap matters, not the support size.
- ③ $p=2$ is impossible because of energy identity $\Rightarrow$ loss of $2^{j/2}$

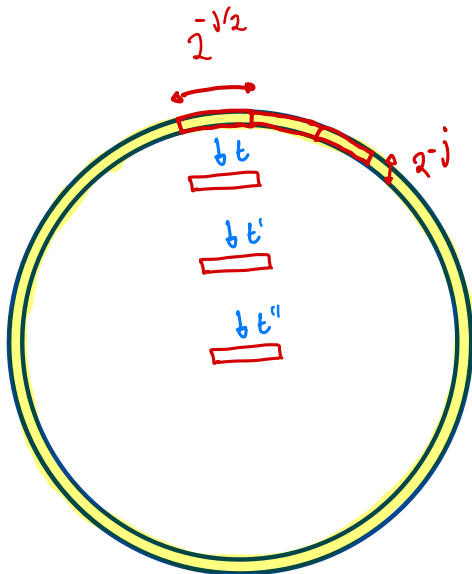
Local smoothing estimates

Let $\{t_\mu\}$ a set of 2^{-j} -separated points in $[1, 2]$

Sogge's question (1991): can we have

$$\left(\sum_{\mu} \| e^{it_{\mu}\sqrt{-\Delta}} f_j \|_{L^p(\mathbb{R}^d)}^p \right)^{1/p} \lesssim 2^{js_p} \| f \|_{L^p(\mathbb{R}^d)}$$

for some large p , where $s_p = (d-1)\left(\frac{1}{2} - \frac{1}{p}\right)$?



$$\frac{\left(\sum_{t_{\mu}} \| e^{it_{\mu}\sqrt{-\Delta}} g_j \|^p_p \right)^{1/p}}{\| g_j \|^p_p} \gtrsim (\# \{t_{\mu}\})^{1/p} \gtrsim 2^{j/p}$$

$$s_p \geq \frac{1}{p} \iff p \geq \frac{2d}{d-1}$$

interpolation!

Most common formulation

Does it hold

$$\left(\int_1^2 \| e^{it\sqrt{-\Delta}} f_j \|_p^p dt \right)^{1/p} \leq 2^{j s_p - j/p} \| f \|_p, \quad p \geq \frac{2d}{d-1} ?$$

Most common formulation

Does it hold

$$\left(\int_1^2 \| e^{it\sqrt{-\Delta}} f_j \|_p^p dt \right)^{1/p} \lesssim 2^{j s_p - j/p} \| f \|_p, \quad p \geq \frac{2d}{d-1} ?$$

U.P. $\int_{t_\mu}^{t_\mu + 2^{-j}} \| e^{it\sqrt{-\Delta}} f_j \|_{L^p(\mathbb{R}^d)}^p dt \approx 2^{-j} \| e^{it_\mu \sqrt{-\Delta}} f_j \|_{L^p(\mathbb{R}^d)}^p$

Most common formulation

Does it hold

$$\left(\int_1^2 \| e^{it\sqrt{-\Delta}} f_j \|_p^p dt \right)^{1/p} \lesssim 2^{j s_p - j/p} \| f \|_p, \quad p \geq \frac{2d}{d-1} ?$$

U.P. $\int_{t_\mu}^{t_\mu + 2^{-j}} \| e^{it\sqrt{-\Delta}} f_j \|_{L^p(\mathbb{R}^d)}^p dt \approx 2^{-j} \| e^{it_\mu \sqrt{-\Delta}} f_j \|_{L^p(\mathbb{R}^d)}^p$

State of the art: $d=2$ Guth-Wang-Zhang (2020) solved $2^{j\varepsilon}$ -loss

Most common formulation

Does it hold

$$\left(\int_1^2 \| e^{it\sqrt{-\Delta}} f_j \|_p^p dt \right)^{1/p} \lesssim 2^{j s_p - j/p} \| f \|_p, \quad p \geq \frac{2d}{d-1} ?$$

U.P. $\int_{t_\mu}^{t_\mu + 2^{-j}} \| e^{it\sqrt{-\Delta}} f_j \|_{L^p(\mathbb{R}^d)}^p dt \approx 2^{-j} \| e^{it_\mu \sqrt{-\Delta}} f_j \|_{L^p(\mathbb{R}^d)}^p$

State of the art: $d=2$ Guth-Wang-Zhang (2020) solved $2^{j\varepsilon}$ -loss

$d \geq 3$ Bourgain-Demeter (2015)

$$p \geq \frac{2(d+1)}{d-1}$$

Most common formulation

Does it hold

$$\left(\int_1^2 \| e^{it\sqrt{-\Delta}} f_j \|_p^p dt \right)^{1/p} \lesssim 2^{j s_p - j/p} \| f \|_p, \quad p \geq \frac{2d}{d-1} ?$$

U.P. $\int_{t_\mu}^{t_\mu + 2^{-j}} \| e^{it\sqrt{-\Delta}} f_j \|_{L^p(\mathbb{R}^d)}^p dt \approx 2^{-j} \| e^{it_\mu \sqrt{-\Delta}} f_j \|_{L^p(\mathbb{R}^d)}^p$

State of the art: $d=2$ Guth-Wang-Zhang (2020) solved $2^{j\varepsilon}$ -loss

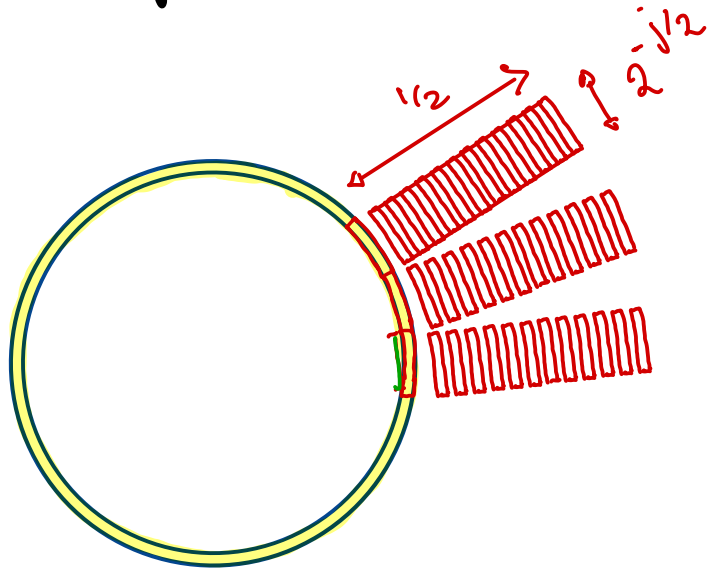
$d \geq 3$ Bourgain-Demeter (2015)

$$p \geq \frac{2(d+1)}{d-1}$$

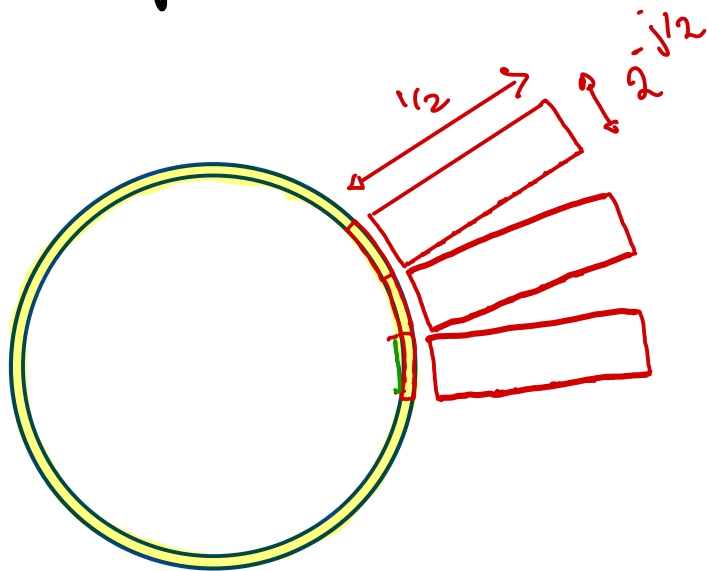
Gan-He-Li-Wu (2026)

$$p \geq \begin{cases} 2 + \frac{8}{3d-3} & \text{if } d \text{ is odd} \\ 2 + \frac{8}{3d-2} O_d\left(\frac{1}{\cdot}\right) & \text{if } d \text{ is even} \end{cases}$$

KaKeya example

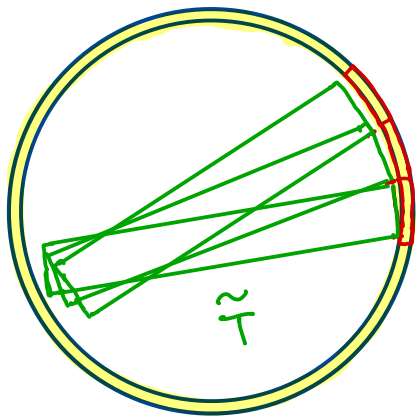


Kakeya example



$$f_j = \sum_{T \in \mathcal{T}_j} \mathbb{1}_T \Rightarrow \|f_j\|_p \sim \left| \bigcup_{T \in \mathcal{T}_j} T \right|^{1/p}$$

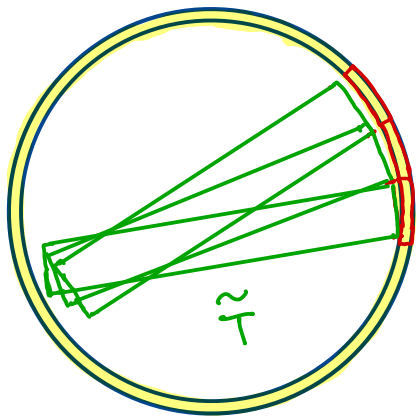
Kakeya example



$$f_j = \sum_{T \in \mathcal{T}_j} \mathbb{1}_T \Rightarrow \|f_j\|_p \sim \left| \bigcup_{T \in \mathcal{T}_j} T \right|^{1/p}$$

$$e^{it\sqrt{-\Delta}} f_j \text{ supported on } \bigcup_{T \in \mathcal{T}_j} T \text{ for } 1 \leq t \leq 2$$

Kakeya example

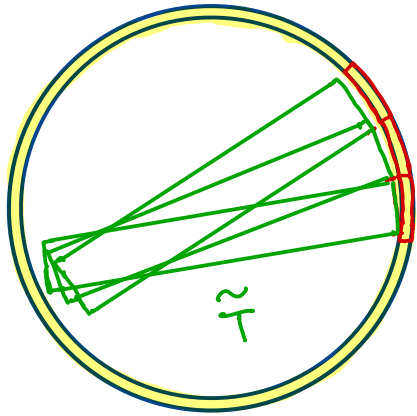


$$f_j = \sum_{T \in \mathcal{T}_j} \mathbb{1}_T \Rightarrow \|f_j\|_p \sim \left| \bigcup_{T \in \mathcal{T}_j} T \right|^{1/p}$$

$$e^{it\sqrt{-\Delta}} f_j \text{ supported on } \bigcup_{T \in \mathcal{T}_j} \tilde{T} \text{ for } 1 \leq t \leq 2$$

$$\|f_j\|_2 = \|e^{it\sqrt{-\Delta}} f_j\|_{L^2(\mathbb{R}^d \times [1,2])} \stackrel{\text{H\"older}}{\leq} \|e^{it\sqrt{-\Delta}} f_j\|_{L^p(\mathbb{R}^d \times [1,2])} \left| \bigcup_{T \in \mathcal{T}_j} \tilde{T} \right|^{\frac{1}{2} - \frac{1}{p}}$$

Kakeya example



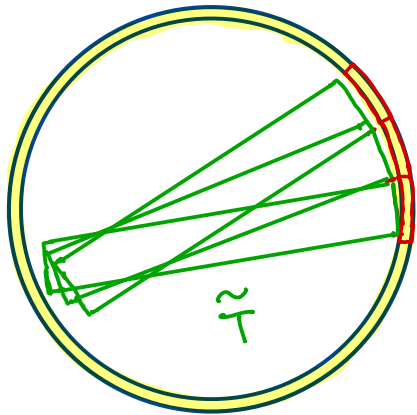
$$f_j = \sum_{T \in \mathcal{T}_j} \mathbb{1}_T \Rightarrow \|f_j\|_p \sim \left| \bigcup_{T \in \mathcal{T}_j} T \right|^{1/p}$$

$$e^{it\sqrt{-\Delta}} f_j \text{ supported on } \bigcup_{T \in \mathcal{T}_j} \tilde{T} \text{ for } 1 \leq t \leq 2$$

$$\|f_j\|_2 = \|e^{it\sqrt{-\Delta}} f_j\|_{L^2(\mathbb{R}^d \times [1,2])} \stackrel{\text{H\"older}}{\leq} \|e^{it\sqrt{-\Delta}} f_j\|_{L^p(\mathbb{R}^d \times [1,2])} \left| \bigcup_{T \in \mathcal{T}_j} \tilde{T} \right|^{\frac{1}{2} - \frac{1}{p}}$$

$$\frac{\|e^{it\sqrt{-\Delta}} f_j\|_{L^p(\mathbb{R}^d \times [1,2])}}{\|f_j\|_{L^p(\mathbb{R}^d)}} \gtrsim \frac{\left| \bigcup_{T \in \mathcal{T}_j} T \right|^{\frac{1}{2} - \frac{1}{p}}}{\left| \bigcup_{T \in \mathcal{T}_j} \tilde{T} \right|^{\frac{1}{2} - \frac{1}{p}}}$$

Kakeya example



$$f_j = \sum_{T \in \mathbb{T}_j} \mathbb{1}_T \Rightarrow \|f_j\|_p \sim \left| \bigcup_{T \in \mathbb{T}_j} T \right|^{1/p}$$

$$e^{it\sqrt{-\Delta}} f_j \text{ supported on } \bigcup_{T \in \mathbb{T}_j} \tilde{T} \text{ for } 1 \leq t \leq 2$$

$$\|f_j\|_2 = \|e^{it\sqrt{-\Delta}} f_j\|_{L^2(\mathbb{R}^d \times [1,2])} \stackrel{\text{H\"older}}{\leq} \|e^{it\sqrt{-\Delta}} f_j\|_{L^p(\mathbb{R}^d \times [1,2])} \left| \bigcup_{T \in \mathbb{T}_j} \tilde{T} \right|^{\frac{1}{2} - \frac{1}{p}}$$

$$\frac{\|e^{it\sqrt{-\Delta}} f_j\|_{L^p(\mathbb{R}^d \times [1,2])}}{\|f_j\|_{L^p(\mathbb{R}^d)}} \gtrsim \frac{\left| \bigcup_{T \in \mathbb{T}_j} T \right|^{\frac{1}{2} - \frac{1}{p}}}{\left| \bigcup_{T \in \mathbb{T}_j} \tilde{T} \right|^{\frac{1}{2} - \frac{1}{p}}}$$

$$\gtrsim j^{\frac{1}{2} - \frac{1}{p}}$$

→ existence of Kakeya sets

Fractal local smoothing for wave equation?

Question : given $E \in [1, 2]$, $2 \leq p < \infty$, for what values of s can we have

$$\left(\sum_{t \in E_j} \| e^{it\sqrt{-\Delta}} f_j \|_{L^p(\mathbb{R}^d)}^p dt \right)^{1/p} \leq 2^{js} \| f \|_{L^p(\mathbb{R}^d)} ?$$

Here $E_j \equiv 2^{-j}$ -separated points in E

Fractal local smoothing for wave equation?

Question : given $E \subseteq [1, 2]$, $2 \leq p < \infty$, for what values of s can we have

$$\left(\sum_{t \in E_j} \|e^{it\sqrt{-\Delta}} f_j\|_{L^p(\mathbb{R}^d)}^p dt \right)^{1/p} \leq 2^{js} \|f\|_{L^p(\mathbb{R}^d)} ?$$

Here $E_j \equiv 2^{-j}$ -separated points in E

- Previous examples give
$$s \geq s_p \quad \text{and} \quad 2^{js} \geq N(E, 2^{-j})^{1/p}$$

intervals of length 2^{-j} needed to cover $E \subseteq [1, 2]$

Fractal local smoothing for wave equation?

Question : given $E \subseteq [1, 2]$, $2 \leq p < \infty$, for what values of s can we have

$$\left(\sum_{t \in E_j} \|e^{it\sqrt{-\Delta}} f_j\|_{L^p(\mathbb{R}^d)}^p dt \right)^{1/p} \leq 2^{js} \|f\|_{L^p(\mathbb{R}^d)} ?$$

Here $E_j \equiv 2^{-j}$ -separated points in E

- Previous examples give $s \geq s_p$ and $2^{js} \geq N(E, 2^{-j})^{1/p}$ $\nearrow$ # intervals of length 2^{-j} needed to cover $E \subseteq [1, 2]$

- Interesting range is $2 < p < \frac{2d}{d-1}$

Fractal local smoothing for wave equation?

Question: given $E \subseteq [1, 2]$, $2 \leq p < \infty$, for what values of s can we have

$$\left(\sum_{t \in E_j} \|e^{it\sqrt{-\Delta}} f_j\|_{L^p(\mathbb{R}^d)}^p dt \right)^{1/p} \leq 2^{js} \|f\|_{L^p(\mathbb{R}^d)} \quad ?$$

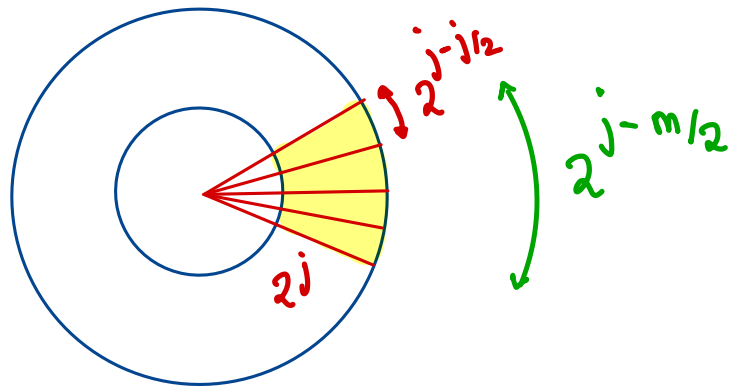
Here $E_j \equiv 2^{-j}$ -separated points in E

- Previous examples give $s \geq s_p$ and $2^{js} \geq N(E, 2^{-j})^{1/p}$ $\uparrow$ # intervals of length 2^{-j} needed to cover $E \subseteq [1, 2]$
- Interesting range is $2 < p < \frac{2d}{d-1}$
- But Minkowski dimension alone is not enough! Multi-scale structure of E !

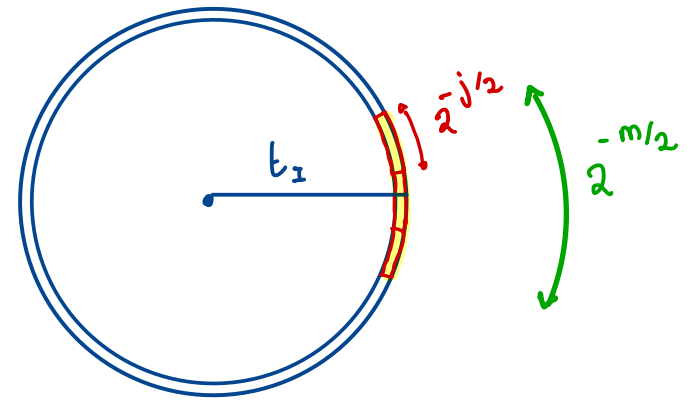
Focusing example for fractal sets : intermediate sectors

$$I = [t_I, s_I] \subseteq [1, 2], \quad 2^{-j} \lesssim |I| \approx 2^{-j+m}, \quad 0 \leq m \leq j$$

ξ -space



x -space

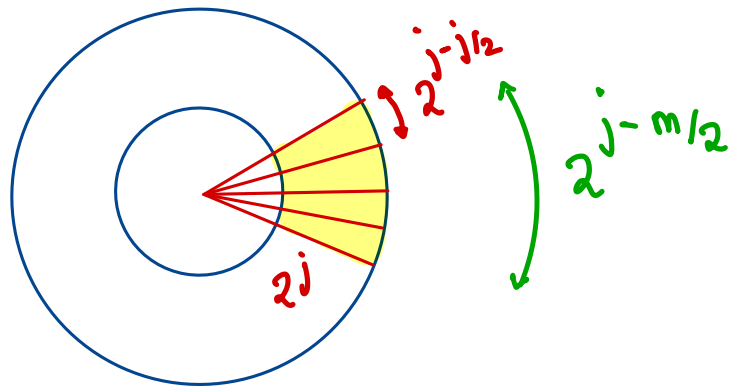


$$\hat{f}_{j,I}(\xi) := e^{-it_I|\xi|} \chi(2^{-j}\xi) \psi(2^{m/2}(\frac{\xi}{|\xi|} - e_1))$$

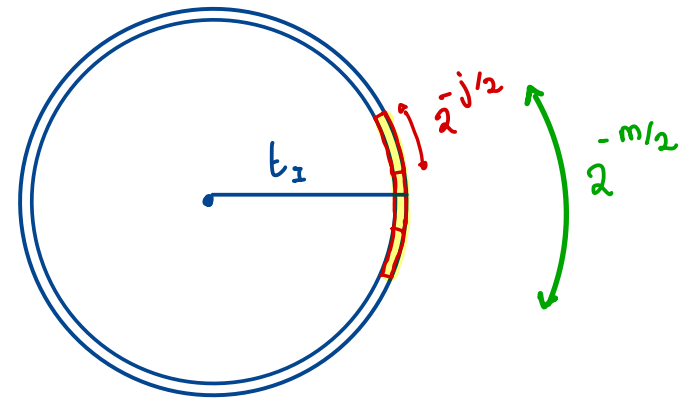
Focusing example for fractal sets : intermediate sectors

$$I = [t_I, s_I] \subseteq [1, 2], \quad 2^{-j} \lesssim |I| \approx 2^{-j+m}, \quad 0 \leq m \leq j$$

ξ -space



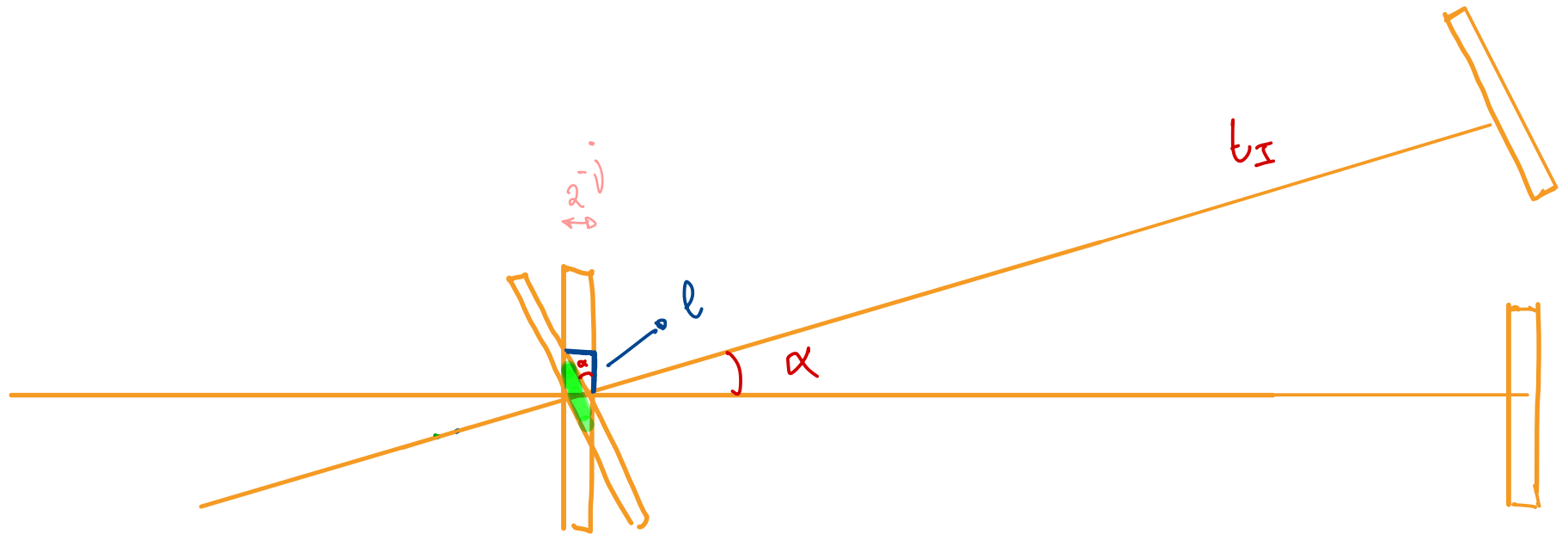
x -space



$$\hat{f}_{j,I}(\xi) := e^{-it_I|\xi|} \chi(2^{-j}\xi) \psi(2^{m/2}(\frac{\xi}{|\xi|} - e_1))$$

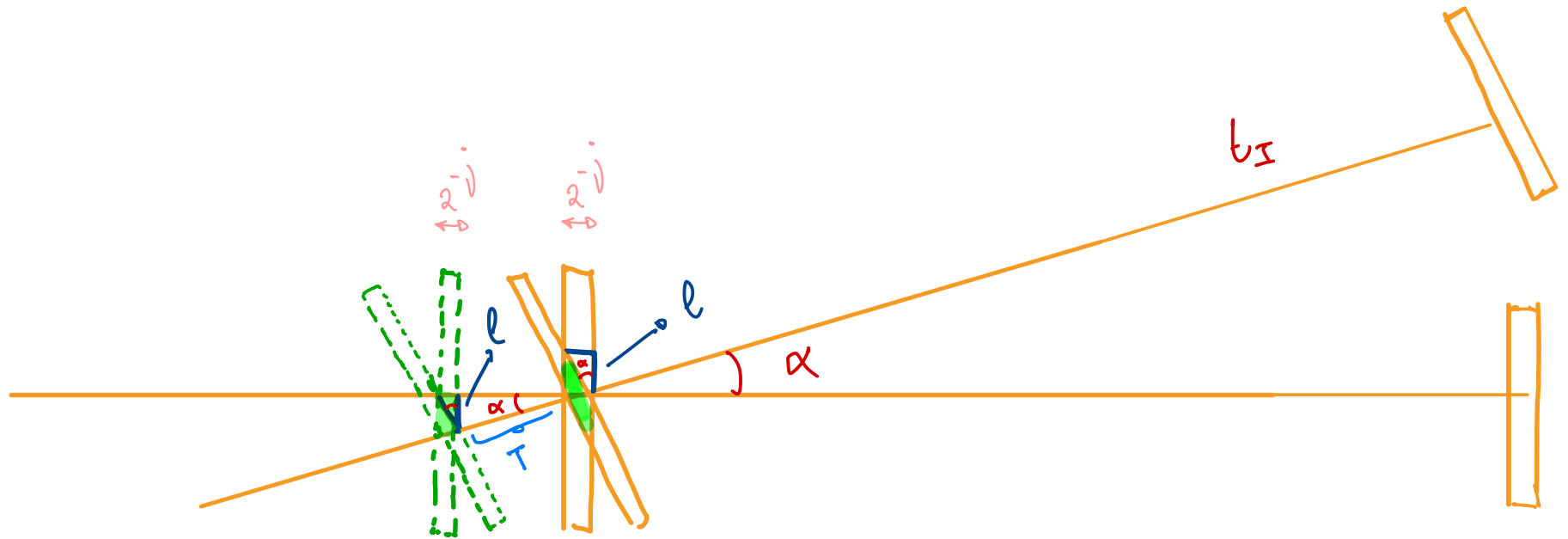
$$\| \hat{f}_{j,I} \|_p \lesssim \underbrace{2^{j \frac{(d+1)}{2}}}_{\text{magnitude of } f \text{ on a small sector}} \underbrace{\left(2^{-j \frac{(d+1)}{2}} \right)^{\frac{1}{p}}}_{\text{volume small sector}} \underbrace{\left(\left(2^{j/2} 2^{-m/2} \right)^{d-1} \right)^{\frac{1}{p}}}_{\substack{\# \text{ small} \\ \text{sectors} \\ \text{in big sector}}}$$

Focusing example for fractal sets : intermediate sectors

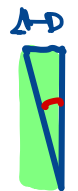


2^{-j}
 $l \approx \frac{2^{-j}}{\sin \alpha} \approx \frac{2^{-j}}{\alpha} \approx 2^{-j+m/2}$

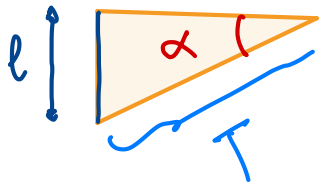
Focusing example for fractal sets : intermediate sectors



2^{-j}

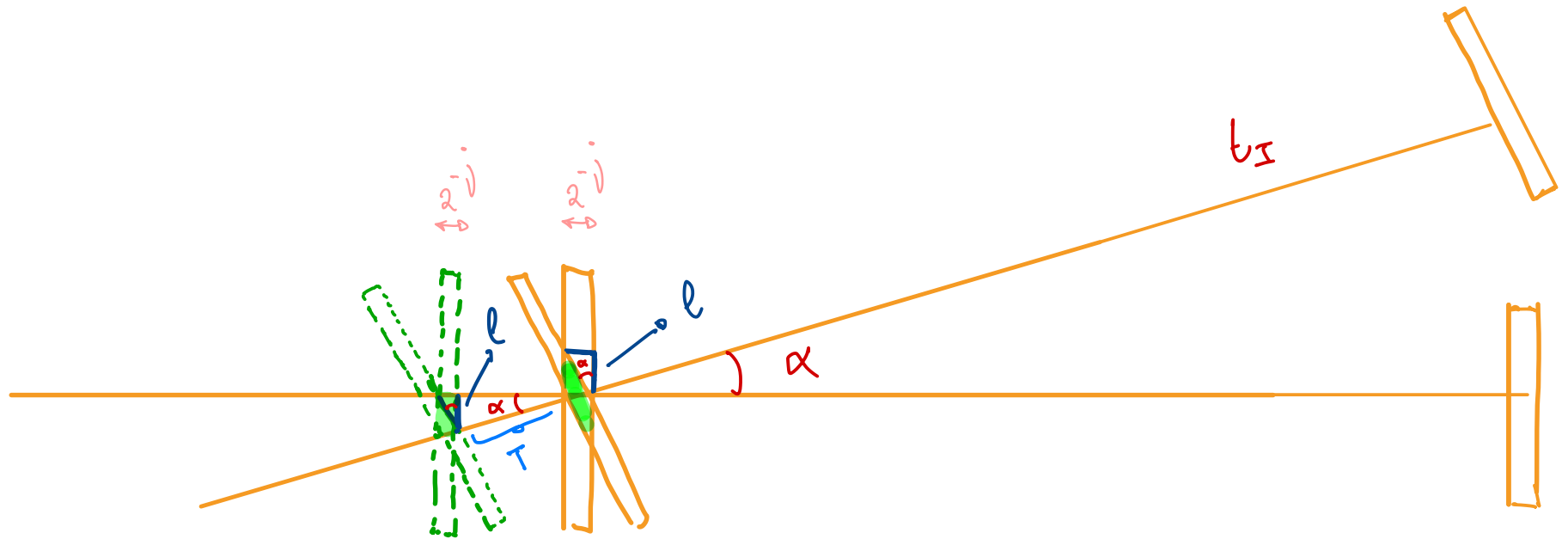


$$l \approx \frac{2^{-j}}{\sin \alpha} \approx \frac{2^{-j}}{\alpha} \approx 2^{-j+m/2}$$

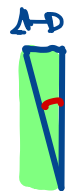


$$T \approx \frac{l}{\sin \alpha} \approx \frac{l}{\alpha} \approx 2^{-j+m}$$

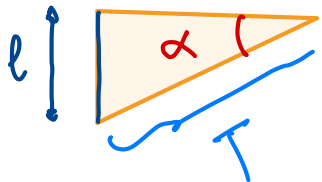
Focusing example for fractal sets : intermediate sectors



2^{-j}



$$l \approx \frac{2^{-j}}{\sin \alpha} \approx \frac{2^{-j}}{\alpha} \approx 2^{-j+m/2}$$



$$T \approx \frac{l}{\sin \alpha} \approx \frac{l}{\alpha} \approx 2^{-j+m} = |I|$$

impose this to keep overlap
for all $t \in I = [t_I, s_I]$

Focusing example for fractal sets : intermediate sectors

$$\left(\sum_{t \in E_j \cap I} \| e^{it\sqrt{-\Delta}} f_{j,I} \|_{L^p(\mathbb{R}^d)}^p \right)^{1/p} \gtrsim$$

$$\gtrsim \left(\sum_{t \in E_j \cap I} \| e^{it\sqrt{-\Delta}} f_{j,I} \|_{L^p(\text{green}_t)}^p \right)^{1/p}$$

Focusing example for fractal sets : intermediate sectors

$$\left(\sum_{t \in E_j \cap I} \| e^{it\sqrt{-\Delta}} f_{j,I} \|_{L^p(\mathbb{R}^d)}^p \right)^{1/p} \gtrsim$$

$$\gtrsim \left(\sum_{t \in E_j \cap I} \| e^{it\sqrt{-\Delta}} f_{j,I} \|_{L^p(\text{green}_t)}^p \right)^{1/p}$$

$$\gtrsim N(E_j \cap I, 2^{-j})^{1/p} \left[\underbrace{2^{-j} 2^{(j+m/2)(d-1)}}_{\text{volume of green}_t} \right]^{1/p}$$

$2^{j(d+1)/2}$
magnitude
on each
small piece

$\left(2^{-\frac{m}{2}} 2^{j/2} \right)^{d-1}$
of pieces
in an $2^{-m/2}$ arc

Focusing example for fractal sets : intermediate sectors

$$\left(\sum_{t \in E_j \cap I} \| e^{it\sqrt{-\Delta}} f_{j,I} \|_{L^p(\mathbb{R}^d)}^p \right)^{1/p} \gtrsim$$

$$\gtrsim \left(\sum_{t \in E_j \cap I} \| e^{it\sqrt{-\Delta}} f_{j,I} \|_{L^p(\text{green}_t)}^p \right)^{1/p}$$

$$\gtrsim N(E_j \cap I, 2^{-j})^{1/p} \underbrace{\left[2^{-j} 2^{(j+m/2)(d-1)} \right]^{1/p}}_{\text{volume of green}_t}$$

$$\underbrace{2^{j(d+1)/2}}_{\text{magnitude on each small piece}} \underbrace{\left(2^{-\frac{m}{2}} 2^{j/2} \right)^{d-1}}_{\substack{\# \text{ of pieces} \\ \text{in an } 2^{-m/2} \text{ arc}}}$$

Recall $\| f_{j,I} \|_p \lesssim \underbrace{2^{j \frac{(d+1)}{2}}}_{\text{magnitude of } f \text{ on a small sector}} \underbrace{\left(2^{-j \frac{(d+1)}{2}} \right)^{1/p}}_{\text{volume small sector}} \underbrace{\left(\left(2^{j/2} 2^{-m/2} \right)^{d-1} \right)^{1/p}}_{\substack{\# \text{ small} \\ \text{sectors} \\ \text{in big sector}}}$

Focusing example for fractal sets : intermediate sectors

$$\left(\sum_{t \in E_j \cap I} \| e^{it\sqrt{-\Delta}} f_{j,I} \|_{L^p(\mathbb{R}^d)}^p \right)^{1/p} \gtrsim$$

$$\gtrsim \left(\sum_{t \in E_j \cap I} \| e^{it\sqrt{-\Delta}} f_{j,I} \|_{L^p(\text{green } t)}^p \right)^{1/p}$$

$$\gtrsim N(E_j \cap I, 2^{-j})^{1/p} \underbrace{\left[2^{-j} 2^{(j+m/2)(d-1)} \right]}_{\text{volume of green } t}^{1/p}$$

$$\underbrace{2^{j(d+1)/2}}_{\text{magnitude on each small piece}}$$

$$\underbrace{\left(2^{-\frac{m}{2}} 2^{j/2} \right)^{d-1}}_{\substack{\# \text{ of pieces} \\ \text{in an } 2^{-m/2} \text{ arc}}}$$

Recall

$$\| f_{j,I} \|_p \lesssim$$

$$\underbrace{2^{j \frac{(d+1)}{2}}}_{\text{magnitude of } f \text{ on a small sector}}$$

$$\underbrace{\left(2^{-j \frac{(d+1)}{2}} \right)^{\frac{1}{p}}}_{\text{volume small sector}}$$

$$\underbrace{\left(2^{j/2} 2^{-m/2} \right)^{d-1}}_{\substack{\# \text{ small} \\ \text{sectors} \\ \text{in big sector}}}^{\frac{1}{p}}$$

Focusing example for fractal sets : intermediate sectors

$$\frac{\left(\sum_{t \in E_j \cap I} \| e^{it\sqrt{-\Delta}} f_{j, \pm} \|_{L^p(\mathbb{R}^d)}^p \right)^{1/p}}{\| f_{j, \pm} \|_p} \gtrsim N(E \cap I, 2^{-j})^{1/p} (2^{-j+m})^{\frac{(d-1)}{2} (\frac{2}{p} - 1)}$$

Focusing example for fractal sets : intermediate sectors

$$\begin{aligned}
 & \frac{\left(\sum_{t \in E_j \cap I} \| e^{it\sqrt{-\Delta}} f_{j, I} \|_{L^p(\mathbb{R}^d)}^p \right)^{1/p}}{\| f_{j, I} \|_p} \gtrsim N(E \cap I, 2^{-j})^{1/p} \underbrace{(2^{-j+m})^{\frac{(d-1)}{2}(\frac{2}{p}-1)}}_{|I|} \\
 & = N(E \cap I, 2^{-j})^{1/p} |I|^{-\underbrace{(d-1)(\frac{1}{2}-\frac{1}{p})}_{=s_p}}
 \end{aligned}$$

Focusing example for fractal sets : intermediate sectors

$$\begin{aligned}
 & \frac{\left(\sum_{t \in E_j \cap I} \| e^{it\sqrt{-\Delta}} f_{j,I} \|_{L^p(\mathbb{R}^d)}^p \right)^{1/p}}{\| f_{j,I} \|_p} \gtrsim N(E \cap I, 2^{-j})^{1/p} \underbrace{(2^{-j+m})^{\frac{(d-1)}{2}(\frac{2}{p}-1)}}_{|I|} \\
 & = N(E \cap I, 2^{-j})^{1/p} |I|^{-\underbrace{(d-1)(\frac{1}{2}-\frac{1}{p})}_{=s_p}}
 \end{aligned}$$

And this happens for all $I \subseteq [1, 2]$

multi-scale
structure !!

Fractal local smoothing conjecture

Given $E \subseteq [1, 2]$, $\alpha > 0$ define

$$\mathcal{L}_E^\#(\alpha) := \limsup_{\delta \rightarrow 0} \frac{\log \left(\sup_{\delta \leq |I| \leq 1} |I|^{-\alpha} N(E \cap I, \delta) \right)}{\log(1/\delta)}$$

Conjecture (B.-Roos-Rutar-Seeger)

For $p \geq 2$, $E \subseteq [1, 2]$, the inequality

$$\left(\sum_{t \in E_j} \| e^{it\sqrt{-\Delta}} f_j \|_{L^p(\mathbb{R}^d)}^p dt \right)^{1/p} \lesssim 2^{js} \| f \|_{L^p(\mathbb{R}^d)}$$

holds for all $s > \frac{\mathcal{L}_E^\#(pS_p)}{p}$, $j \geq 1$.

Fractal local smoothing conjecture

Given $E \subseteq [1, 2]$, $\alpha \geq 0$ define

$$\mathcal{L}_E^\#(\alpha) := \limsup_{\delta \rightarrow 0} \frac{\log \left(\sup_{\delta \leq |I| \leq 1} |I|^{-\alpha} N(E \cap I, \delta) \right)}{\log(1/\delta)}$$

Conjecture (B.-Roos-Rutar-Seeger)

For $p \geq 2$, $E \subseteq [1, 2]$, the inequality

$$\left(\sum_{t \in E_j} \| e^{it\sqrt{-\Delta}} f_j \|_{L^p(\mathbb{R}^d)}^p dt \right)^{1/p} \lesssim 2^{js} \| f \|_{L^p(\mathbb{R}^d)}$$

holds for all $s > \frac{\mathcal{L}_E^\#(pS_p)}{p}$, $j \geq 1$.

Other variants formulated by Ham-Ko-Lee, Wheeler, Lee-Roncal-Zhang-Zhou

Positive evidence

Thm (B.-Roe-Rutar-Seeger)

- If $2 < p < \infty$, true for radical functions and all $E \in [1, 2]$
- If $\boxed{p=2}$, there is an $L^2 \rightarrow L^q$ version for $2 < q < \infty$
which is true for all $E \in [1, 2]$. (TT*-argument)

Positive evidence

Thm (B.-Roe-Rutar-Seeger)

- If $2 < p < \infty$, true for radical functions and all $E \in [1, 2]$
- If $\boxed{p=2}$, there is an $L^2 \rightarrow L^q$ version for $2 < q < \infty$ which is true for all $E \in [1, 2]$. (TT*-argument)

Lee-Roncal-Zhang-Zhao : partial results in $L^p \rightarrow L^q$ category under assumptions on $E \in [1, 2]$.
(bilinear restriction).

Understanding $\lambda_E^\#(\alpha)$

- $\lambda_E^\#$ can be written in terms of the Assouad spectrum of E
- $\lambda_E^\#(\alpha) = \alpha$ if $\alpha \geq \gamma$, where $\gamma = \dim_{\mathbb{R}^A} E$
- In our case $\frac{1}{p} \lambda_E^\#(ps_p) = s_p$ for $p \geq p_\gamma := \frac{2(d-1+\gamma)}{d-1}$
- p_γ is a "fake" analogue of $p = \frac{2d}{d-1}$ for arbitrary E

$\frac{1}{p} \lambda_E^\#(ps_p)$ is not necessarily the interpolated exponent between $2 \leq p \leq p_\gamma$

¡ GRACIAS !

From the Quasi-Geostrophic Shallow-Water Model to the 2D Euler Equations

Two convergence results

Marc Magaña Centelles

Universitat Autònoma de Barcelona

EARCO XXIII — Logroño
29 May 2026

Introduction

Motivation: From QGSW to Euler

QGSW

$$\begin{cases} \partial_t \omega_\lambda + u_\lambda \cdot \nabla \omega_\lambda = 0, \\ u_\lambda = \nabla^\perp (\lambda - \Delta)^{-1} \omega_\lambda, \\ \omega_\lambda|_{t=0} = \omega_{0,\lambda}. \end{cases}$$

- ω_λ : potential vorticity
- u_λ : velocity field
- $\sqrt{\lambda}$ = inverse Rossby radius

Euler

$$\begin{cases} \partial_t \Omega + v \cdot \nabla \Omega = 0, \\ v = \nabla^\perp (-\Delta)^{-1} \Omega, \\ \Omega|_{t=0} = \Omega_0. \end{cases}$$

- Ω : vorticity
- v : velocity field

$$\nabla^\perp = (-\partial_2, \partial_1)^t, \quad \operatorname{div} u_\lambda = \operatorname{div} v = 0$$

Singular limit problem

Does $\omega_\lambda \rightarrow \Omega$ rigorously as $\lambda \rightarrow 0$? In which topology?

Euler equations

- Classical well-posedness theory: Gyunter (1926), Lichtenstein (1927), Wolibner (1933), Kato (1967), Chemin (1998), Constantin (2001), Majda and Bertozzi (2002)...
- Yudovich (1963): global weak solutions for $\Omega_0 \in L^1 \cap L^\infty$.
- *Vortex patches* and *V-states*: Kirchhoff (1876), [Deem and Zabusky \(1978\)](#), Burbea (1982), Hmidi, Mateu, and Verdera (2013), ...

QGSW equations

- Vortex patches and V-states: [Polvani \(1988\)](#), [Polvani, Zabusky, and Flierl \(1989\)](#), [Plotka and Dritschel \(2012\)](#), Dritschel, Hmidi, and Renault (2019), Hmidi and Roulley (2021), Roulley (2023),...

[Blue: numerical studies](#)

Overview of the convergence results

Theorem (Hölder regime ($0 < \gamma < 1$), M.–Mateu–Orobitg, 2026)

$\omega_{0,\lambda} = \Omega_0 \in C_c^\gamma(\mathbb{R}^2)$. *There exists $T > 0$ such that*

$$\lim_{\lambda \rightarrow 0} \sup_{t \in [0, T]} \|\omega_\lambda(\cdot, t) - \Omega(\cdot, t)\|_{C^\gamma} = 0.$$

Theorem (Yudovich regime, Houamed - M. 2025)

$\omega_{0,\lambda} = \Omega_0 \in L^1 \cap L^\infty(\mathbb{R}^2)$, *uniformly in $\lambda > 0$. Then, for any $t \geq 0$, and $p \in (1, \infty)$:*

$$\lim_{\lambda \rightarrow 0} \left(\|\nabla^\perp \Delta^{-1}(\omega_\lambda - \Omega)\|_{L^\infty([0, t]; L^2 \cap L^\infty(\mathbb{R}^2))} + \|\omega_\lambda - \Omega\|_{L^\infty([0, t]; L^p(\mathbb{R}^2))} \right) = 0,$$

If Ω_0 is continuous then for any $t > 0$, $\lim_{\lambda \rightarrow 0} \|\omega_\lambda - \Omega\|_{L^\infty([0, t]; L^\infty(\mathbb{R}^2))} = 0$.

Result 1: Convergence in Hölder Spaces

Preliminaries: Hölder and little Hölder spaces

Let

$$\|f\|_{\mathcal{C}^\gamma} = \|f\|_{L^\infty} + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\gamma}.$$

Definition

The **Hölder space** $\mathcal{C}^\gamma$ consists of all f with $\|f\|_{\mathcal{C}^\gamma} < \infty$.

The **little Hölder space** c^γ is the subset of $\mathcal{C}^\gamma$ satisfying

$$\lim_{h \rightarrow 0} \sup_{0 < |x - y| < h} \frac{|f(x) - f(y)|}{|x - y|^\gamma} = 0.$$

Key property: c^γ is the closure of C^∞ in the norm $\|\cdot\|_{\mathcal{C}^\gamma}$.

Result 1: Statement

Theorem (Convergence in c^γ , M.–Mateu–Orobitg, 2026)

$\omega_{0,\lambda} = \Omega_0 \in c_c^\gamma(\mathbb{R}^2; \mathbb{R})$. for $0 < \gamma < 1$. Then there exist unique solutions $\omega_\lambda(\cdot, t)$, $\Omega(\cdot, t)$ in $c_c^\gamma(\mathbb{R}^2, \mathbb{R})$ and a time $T > 0$ such that

$$\lim_{\lambda \rightarrow 0} \sup_{t \in [0, T]} \|\omega_\lambda(\cdot, t) - \Omega(\cdot, t)\|_{C^\gamma} = 0.$$

Remarks:

- Also true when the initial data for QGSW and Euler are different but sufficiently close.
- Existence and uniqueness for QGSW (and Euler) hold in the classical Hölder space C^γ for all time. (Euler: Majda and Bertozzi (2002), QGSW: M.–Mateu–Orobitg, (2026))

Sketch of the proof (1/3): Lagrangian formulation

- Velocity from vorticity via convolution kernel:

$$u_\lambda = \mathcal{K}_\lambda * \omega_\lambda, \quad \mathcal{K}_\lambda(x) = \frac{\sqrt{\lambda}}{2\pi} \frac{x^\perp}{|x|} K_1(\sqrt{\lambda}|x|), \quad \mathcal{K}_0(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2}.$$

- Flow map $X_\lambda(\alpha, t)$: $\frac{dX_\lambda}{dt} = u_\lambda(X_\lambda, t)$, $X_\lambda(\alpha, 0) = \alpha$.
- Using $\operatorname{div} u_\lambda = 0 \Rightarrow \det(\nabla X_\lambda) = 1$ and

$$\omega_\lambda(x, t) = \omega_0(X_\lambda^{-1}(x, t)), \quad \Omega(x, t) = \omega_0(X_0^{-1}(x, t)).$$

- The flow satisfies the closed ODE

$$\frac{dX_\lambda}{dt} = F_\lambda(X_\lambda), \quad F_\lambda(X)(\alpha) = \int_{\mathbb{R}^2} \mathcal{K}_\lambda(X(\alpha) - X(\alpha')) \omega_0(\alpha') d\alpha'.$$

Question

Does convergence of the flow maps $X_\lambda \rightarrow X_0$ imply convergence of the vorticities $\omega_\lambda \rightarrow \Omega$?

Sketch of the proof (2/3): Uniform well-posedness and flow convergence

- We work in $B = \mathbf{c}^{1,\gamma}(\mathbb{R}^2; \mathbb{R}^2)$ with the open set

$$U_\delta = \{X = \mathbf{e} + \varphi_X : \varphi_X \in \mathbf{c}^{1,\gamma}, \|\varphi_X\|_{\mathbf{c}^{1,\gamma}} < \delta\}, \quad \mathbf{e}(x) = x.$$

For small δ , each $X \in U_\delta$ is bi-Lipschitz.

- The operator F_λ satisfies: $F_\lambda : U_\delta \rightarrow \mathbf{c}^{1,\gamma}$ bounded and locally Lipschitz, **uniformly in λ** .
- Picard–Lindelöf $\Rightarrow$ exists common $T > 0$ and unique flows $X_\lambda \in C([0, T]; U_\delta)$.
- $\sup_{X \in U_\delta} \|F_\lambda(X) - F_0(X)\|_{\mathbf{c}^{1,\gamma}} \xrightarrow{\lambda \rightarrow 0} 0$ + Grönwall = continuous dependence on λ : $X_\lambda \rightarrow X_0$ in $C([0, T]; \mathbf{c}^{1,\gamma})$.
- By Misiólek and Yoneda (2018, Lemma 2.2), we also have $X_\lambda^{-1} \rightarrow X_0^{-1}$ in $\mathbf{c}^{1,\gamma}$.

Sketch of the proof (3/3): Why little Hölder?

Take $\omega_0 \in \mathcal{C}^\gamma$. For any $\eta > 0$ choose smooth ϕ_η with $\|\omega_0 - \phi_\eta\|_{\mathcal{C}^\gamma} < \eta$ (density in $\mathcal{C}^\gamma$).

$$\begin{aligned} \|\omega_\lambda(t) - \Omega(t)\|_{\mathcal{C}^\gamma} &\leq \underbrace{\left\| \omega_0(X_\lambda^{-1}) - \phi_\eta(X_\lambda^{-1}) \right\|_{\mathcal{C}^\gamma}}_{\leq C\|\omega_0 - \phi_\eta\|_{\mathcal{C}^\gamma} < C\eta} + \underbrace{\left\| \phi_\eta(X_\lambda^{-1}) - \phi_\eta(X_0^{-1}) \right\|_{\mathcal{C}^\gamma}}_{\leq C\|\nabla \phi_\eta\|_{\mathcal{C}^\gamma} \|X_\lambda^{-1} - X_0^{-1}\|_{\mathcal{C}^{1,\gamma}}} \\ &\quad + \underbrace{\left\| \phi_\eta(X_0^{-1}) - \omega_0(X_0^{-1}) \right\|_{\mathcal{C}^\gamma}}_{\leq C\|\omega_0 - \phi_\eta\|_{\mathcal{C}^\gamma} < C\eta}. \end{aligned}$$

Since $\eta > 0$ is arbitrary, we obtain $\lim_{\lambda \rightarrow 0} \|\omega_\lambda - \Omega\|_{\mathcal{C}^\gamma} = 0$.

Key point: Density of smooth functions in $\mathcal{C}^\gamma$. (Fails in $\mathcal{C}^\gamma$!)

Result 2: Convergence in Yudovich Class

Result 2: Statement

Theorem (Strong convergence in Yudovich class, Houamed–M., 2025)

Let $\omega_{0,\lambda}, \Omega_0 \in L^1 \cap L^\infty(\mathbb{R}^2)$ uniformly in λ , with

$$\lim_{\lambda \rightarrow 0} \left(\|\nabla^\perp \Delta^{-1}(\omega_{0,\lambda} - \Omega_0)\|_{L^2} + \|\omega_{0,\lambda} - \Omega_0\|_{L^2} \right) = 0.$$

Then for any $t > 0$ and $p \in (1, \infty)$,

$$\lim_{\lambda \rightarrow 0} \left(\|\nabla^\perp \Delta^{-1}(\omega_\lambda - \Omega)\|_{L^\infty([0,t]; L^2 \cap L^\infty(\mathbb{R}^2))} + \|\omega_\lambda - \Omega\|_{L^\infty([0,t]; L^p(\mathbb{R}^2))} \right) = 0,$$

If additionally Ω_0 is continuous and $\|\omega_{0,\lambda} - \Omega_0\|_{L^\infty} \rightarrow 0$, then convergence also holds in L^∞ .

Remark: existence and uniqueness of solutions for all time (Euler: Yudovich (1963), QGSW: M.–Mateu–Oróbitg, (2026))

Core Idea: Extrapolation Compactness Method

A general strategy (Arsénio and Houamed (2024)):

- 1 **Level 1:** Prove convergence in a low-regularity space (slightly better than $B_{2,\infty}^{-1}$).
- 2 **Level 2:** Reduce convergence in L^2 to evanescence of high frequencies:

$$\|\omega_\lambda - \Omega\|_{L^2} \rightarrow 0 \iff \|\mathbf{1}_{|D| \geq \Theta_\lambda} \omega_\lambda\|_{L^2} \rightarrow 0$$

for some $\Theta_\lambda \rightarrow \infty$ with $\Theta_\lambda \|\omega_\lambda - \Omega\|_{X_{2,\infty}^{-1}} \rightarrow 0$.

- 3 **Level 3:** Show high frequencies vanish using a transport equation estimate.

Level 1/3: Convergence in a Low-Regularity Space

Define $X_{2,\infty}^{-1}$: $\|f\|_{X_{2,\infty}^{-1}} = \|S_0 \nabla^\perp \Delta^{-1} f\|_{L^2} + \sup_{j \geq 0} 2^{-j} \|\Delta_j f\|_{L^2}$.

S_0 : low-frequency cut-off, Δ_j : Littlewood–Paley blocks.

Theorem

$$\lim_{\lambda \rightarrow 0} \|\omega_{0,\lambda} - \Omega_0\|_{X_{2,\infty}^{-1}} = 0 \implies \lim_{\lambda \rightarrow 0} \|\omega_\lambda - \Omega\|_{L^\infty([0,t]; X_{2,\infty}^{-1})} = 0.$$

Idea of the proof:

- Set $f_\lambda = \omega_\lambda - \Omega$. It satisfies

$$\partial_t f_\lambda + v \cdot \nabla f_\lambda = (u_\lambda - v) \cdot \nabla \omega_\lambda.$$

- **Low frequencies:** energy estimate on $S_0 \nabla^\perp \Delta^{-1} f_\lambda$.
- **High frequencies:** dyadic estimates + Bony decomposition + commutators.
- Both parts + Osgood's lemma:

$$\|f_\lambda\|_{L^\infty([0,T]; X_{2,\infty}^{-1})} \leq \left(\|f_{0,\lambda}\|_{X_{2,\infty}^{-1}}^2 + C_{**} \lambda T \right)^{\frac{1}{2} \exp(-C_{**} T)}.$$

Level 2/3 – Reduction to High Frequencies

Proposition

$$\lim_{\lambda \rightarrow 0} \|\omega_\lambda - \Omega\|_{L^\infty([0,T];L^2)} = 0 \iff \lim_{\lambda \rightarrow 0} \|\mathbb{1}_{|D| \geq \Theta_\lambda} \omega_\lambda\|_{L^\infty([0,T];L^2)} = 0,$$

for some $\Theta_\lambda > 0$ satisfying

$$\lim_{\lambda \rightarrow 0} \Theta_\lambda = \infty \quad \text{and} \quad \lim_{\lambda \rightarrow 0} \Theta_\lambda \|\omega_\lambda - \Omega\|_{L^\infty([0,T];X_{2,\infty}^{-1})} = 0.$$

Idea: " $\Leftarrow$ " $f \in X_{2,\infty}^{-1} \iff S_0 \nabla^\perp \Delta^{-1} f \in L^2$ and $(\text{Id} - S_0)f \in B_{2,\infty}^{-1}$,

$$\begin{aligned} \|f_\lambda\|_{L^\infty([0,T];L^2)} &\lesssim \underbrace{\left\| S_0 \nabla^\perp \Delta^{-1} f_\lambda \right\|_{L^\infty([0,T];L^2)}}_{\text{Level 1} \implies \rightarrow 0} + \underbrace{\left\| \mathbb{1}_{|D| \geq \Theta_\lambda} \Omega \right\|_{L^\infty([0,T];L^2)}}_{\text{time continuity} \implies \rightarrow 0} \\ &\quad + \underbrace{\left\| \mathbb{1}_{|D| \geq \Theta_\lambda} \omega_\lambda \right\|_{L^\infty([0,T];L^2)} + \Theta_\lambda^2 \|(\text{Id} - S_0)f_\lambda\|_{L^\infty([0,T];B_{2,\infty}^{-1})}^2}_{\text{Level 1} \implies \rightarrow 0}. \end{aligned}$$

Level 3/3: Evanescence of High Frequencies + Conclusion

Theorem

For $\Theta_\lambda \rightarrow \infty$ with $\Theta_\lambda \|f_{0,\lambda}\|_{X_{2,\infty}^{-1}} \rightarrow 0$ and $\Theta_\lambda \sqrt{\lambda} \rightarrow 0$,

$$\lim_{\lambda \rightarrow 0} \|\mathbf{1}_{|D| \geq \Theta_\lambda} \omega_\lambda\|_{L^\infty([0,T]; L^2)} = 0.$$

Key estimate for $\partial_t \omega_\lambda + u_\lambda \cdot \nabla \omega_\lambda = 0$:

$$\frac{d}{dt} \|\text{high freq. of } \omega_\lambda\|_{L^2}^2 \lesssim C \|\text{high freq. of } \omega_\lambda\|_{L^2}^2 + (\text{vanishing terms}).$$

- r.h.s terms comes from the interaction high-high: $\nabla u_\lambda \cdot \omega_\lambda$,
- The “vanishing terms” include: high freq. of initial data and products of high freq. of ∇u_λ with **low** freq. of ω_λ .
- Gronwall's lemma finishes the proof

Conclusion: By Level 2, $\|\omega_\lambda - \Omega\|_{L_t^\infty L^2} \rightarrow 0$; interpolation gives L^p for $1 < p < \infty$.

Summary

Summary and Comparison

Hölder regime

- Initial data: c_c^γ
- Convergence: C^γ
- Method: Lagrangian / flow maps
- Sharpness: ???

Yudovich regime

- Initial data: $L^1 \cap L^\infty$
- Convergence: L^p , $1 < p < \infty$
- Method: Eulerian / Extrapolation compactness
- Sharpness: **YES!**

Theorem (Houamed–M., 2025)

Take $\omega_{0,\lambda} = \Omega_0 = \mathbf{1}_D$ where D is a domain with rectifiable boundary that is **not a disk**. Then

$$\|\omega_\lambda - \Omega\|_{L^\infty((t_1, t_2); L^\infty)} \geq 1 \quad \text{for any } 0 < t_1 < t_2.$$

¡Gracias por vuestra atención!

References I

-  Arsénio, Diogo and Haroune Houamed (2024). “Stability analysis of two-dimensional ideal flows with applications to viscous fluids and plasmas”. In: *Int. Math. Res. Not. IMRN* 2024.8, pp. 7032–7059.
-  Burbea, Jacob (1982). “Motions of vortex patches”. In: *Letters in Mathematical Physics* 6, pp. 1–16.
-  Chemin, Jean-Yves (1998). *Perfect incompressible fluids*. Vol. 14. Oxford Lecture Series in Mathematics and its Applications. Translated from the 1995 French original by Isabelle Gallagher and Dragos Iftimie. The Clarendon Press, Oxford University Press, New York, pp. x+187.
-  Constantin, Peter (2001). “An Eulerian-Lagrangian approach for incompressible fluids: local theory”. In: *J. Amer. Math. Soc.* 14.2, pp. 263–278. DOI: 10.1090/S0894-0347-00-00364-7.

References II



Deem, Gary S. and Norman J. Zabusky (1978). “Vortex waves: Stationary “V states”, interactions, recurrence, and breaking”. In: *Physical Review Letters* 40.13, pp. 859–862.



Dritschel, David G., Taoufik Hmidi, and Coralie Renault (2019). “Imperfect bifurcation for the quasi-geostrophic shallow-water equations”. In: *Arch. Ration. Mech. Anal.* 231.3, pp. 1853–1915. ISSN: 0003-9527,1432-0673. DOI: 10.1007/s00205-018-1312-7. URL: <https://doi.org/10.1007/s00205-018-1312-7>.



Gyunter, Nikolai (1926). “On the motion of a fluid contained in a given moving vessel”. In: *Izvestiya Akademii Nauk SSSR. Seriya Fiziko-Matematicheskaya* 20. Continued in vol. 21 (1927), pp. 621–656, 735–756, 1139–1162; vol. 22 (1928), pp. 9–30. In Russian, pp. 1323–1348, 1503–1532.

References III

-  Hmidi, Taoufik, Joan Mateu, and Joan Verdera (2013). “Boundary regularity of rotating vortex patches”. In: *Archive for Rational Mechanics and Analysis* 209, pp. 171–208.
-  Hmidi, Taoufik and Emeric Roulley (2021). “Time quasi-periodic vortex patches for quasi-geostrophic shallow-water equations”. In: *arXiv preprint arXiv:2110.13751*. To appear in *Mémoires de la Société Mathématique de France*, arXiv:2110.13751.
-  Houamed, Haroune and Marc Magaña (Dec. 2025). “Sharp strong convergence in ideal flows”. In: *Nonlinearity* 38.12, p. 125012. DOI: 10.1088/1361-6544/ae25b0. URL: <https://doi.org/10.1088/1361-6544/ae25b0>.
-  Kato, Tosio (1967). “On classical solutions of the two-dimensional non-stationary Euler equation”. In: *Archive for Rational Mechanics and Analysis* 25.3, pp. 188–200.
-  Kirchhoff, Gustav (1876). *Vorlesungen über mathematische Physik*. Vol. 1. B. G. Teubner.

References IV

-  Lichtenstein, Leon (1927). “Über einige Existenzprobleme der Hydrodynamik: Zweite Abhandlung Nichthomogene, unzusammendrückbare, reibungslose Flüssigkeiten”. In: *Mathematische Zeitschrift* 26.1, pp. 196–323.
-  Magaña, Marc, Joan Mateu, and Joan Orobitg (2026). “The regularity of the boundary of vortex patches for the quasi-geostrophic shallow-water equations”. In: *arXiv preprint arXiv:2602.22767*.
-  Majda, Andrew J. and Andrea L. Bertozzi (2002). *Vorticity and incompressible flow*. Vol. 27. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, pp. xii+545.
-  Misiólek, Gerard and Tsuyoshi Yoneda (2018). “Continuity of the solution map of the Euler equations in Hölder spaces and weak norm inflation in Besov spaces”. In: *Trans. Amer. Math. Soc.* 370.7, pp. 4709–4730. DOI: 10.1090/tran/7101.

-  Płotka, Hanna and David G. Dritschel (2012). “Quasi-geostrophic shallow-water vortex-patch equilibria and their stability”. In: *Geophys. Astrophys. Fluid Dyn.* 106.6, pp. 574–595. ISSN: 0309-1929,1029-0419. DOI: 10.1080/03091929.2012.674128. URL: <https://doi.org/10.1080/03091929.2012.674128>.
-  Polvani, Lorenzo M. (1988). “Geostrophic vortex dynamics”. PhD thesis. Massachusetts Institute of Technology.
-  Polvani, Lorenzo M., Norman J. Zabusky, and Glenn R. Flierl (1989). “Two-layer geostrophic vortex dynamics. Part 1. Upper-layer V-states and merger”. In: *Journal of Fluid Mechanics* 205, pp. 215–242.
-  Roulley, Emeric (2023). “Vortex rigid motion in quasi-geostrophic shallow-water equations”. In: *Asymptot. Anal.* 133.3, pp. 397–446. ISSN: 0921-7134,1875-8576. DOI: 10.3233/asy-221817. URL: <https://doi.org/10.3233/asy-221817>.



Wolibner, Witold (1933). “Un theoreme sur l’existence du mouvement plan d’un fluide parfait, homogène, incompressible, pendant un temps infiniment long”. In: *Mathematische Zeitschrift* 37, pp. 698–726.



Yudovich, Viktor I. (Jan. 1963). “Some bounds for solutions of elliptic equations”. In: *Translations. Series 2. American Mathematical Society*. 56.

Some Results on Vector-Valued Fock Spaces

Óscar Blasco¹ Marc Ventura¹

¹Universitat de València

May 30, 2026

EARCO 2026 - Encuentros de Análisis Real y Complejo



EARCO 2026

Encuentros de Análisis Real y Complejo

- The Bergman space $A^2(\mathbb{D})$ consists of the holomorphic functions on $\mathbb{D}$ such that

$$\|f\|^2 = \alpha \int_{\mathbb{D}} |f(z)|^2 dA(z) = \frac{1}{\pi} \int_{\mathbb{D}} |f(z)|^2 dz < \infty.$$

In this space, we use the normalized area measure on $\mathbb{D}$.

- For Fock spaces, we use the Gaussian measure on $\mathbb{C}$. The Fock space F_1^2 consists of entire functions $f: \mathbb{C} \rightarrow \mathbb{C}$ such that

$$\|f\|_2^2 = \frac{1}{\pi} \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} dA(z) < \infty.$$

where dA is the Lebesgue measure on $\mathbb{C}$.

- The Bergman space $A^2(\mathbb{D})$ consists of the holomorphic functions on $\mathbb{D}$ such that

$$\|f\|^2 = \alpha \int_{\mathbb{D}} |f(z)|^2 dA(z) = \frac{1}{\pi} \int_{\mathbb{D}} |f(z)|^2 dz < \infty.$$

In this space, we use the normalized area measure on $\mathbb{D}$.

- For Fock spaces, we use the Gaussian measure on $\mathbb{C}$. The Fock space F_{α}^2 consists of entire functions $f: \mathbb{C} \rightarrow \mathbb{C}$ such that

$$\|f\|_{2,\alpha}^2 = \frac{\alpha}{\pi} \int_{\mathbb{C}} |f(z)|^2 e^{-\alpha|z|^2} dA(z) < \infty.$$

where dA is the Lebesgue measure on $\mathbb{C}$.

Definition. The spaces F_α^p

For $1 \leq p < \infty$ and $\alpha > 0$, the Fock space F_α^p consists of the entire functions $f: \mathbb{C} \rightarrow \mathbb{C}$ such that

$$\|f\|_{p,\alpha}^p = \frac{\alpha p}{2\pi} \int_{\mathbb{C}} |f(z)|^p e^{-\frac{\alpha p}{2}|z|^2} dA(z) < \infty.$$

Definition. The spaces F_α^p

For $1 \leq p < \infty$ and $\alpha > 0$, the Fock space F_α^p consists of the entire functions $f: \mathbb{C} \rightarrow \mathbb{C}$ such that

$$\|f\|_{p,\alpha}^p = \frac{\alpha p}{2\pi} \int_{\mathbb{C}} |f(z)|^p e^{-\frac{\alpha p}{2}|z|^2} dA(z) < \infty.$$

For $p = \infty$, the space F_α^∞ consists of the entire functions such that

$$\|f\|_{\infty,\alpha} = \sup_{z \in \mathbb{C}} |f(z)| e^{-\frac{\alpha}{2}|z|^2} < \infty.$$

The spaces F_α^p are Banach spaces and, for $p = 2$, are separable Hilbert spaces.

For each measurable function $f : \mathbb{C} \rightarrow \mathbb{C}$, $p > 0$ and $r \geq 0$, we denote

$$M_p(f, r) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})| d\theta \right)^{1/p}, \quad M_\infty(f, r) = \sup_{|z|=r} |f(z)|.$$

For each measurable function $f : \mathbb{C} \rightarrow \mathbb{C}$, $p > 0$ and $r \geq 0$, we denote

$$M_p(f, r) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})| d\theta \right)^{1/p}, \quad M_\infty(f, r) = \sup_{|z|=r} |f(z)|.$$

The norms above can be computed as

$$\|f\|_{p,\alpha}^p = \alpha p \int_0^\infty M_p(f, r)^p r e^{-\frac{\alpha p}{2} r^2} dr, \quad \|f\|_{\infty,\alpha} = \sup_{r \geq 0} M_\infty(f, r) e^{-\frac{\alpha}{2} r^2}.$$

We can also define the space f_α^∞ of entire functions $f : \mathbb{C} \rightarrow \mathbb{C}$ such that

$$\lim_{|z| \rightarrow \infty} |f(z)| e^{-\frac{\alpha}{2}|z|^2} = 0.$$

We can also define the space f_α^∞ of entire functions $f : \mathbb{C} \rightarrow \mathbb{C}$ such that

$$\lim_{|z| \rightarrow \infty} |f(z)| e^{-\frac{\alpha}{2}|z|^2} = 0.$$

The inclusions between the spaces are as follows:

$$F_\alpha^1 \subset F_\alpha^p \subset F_\alpha^q \subset f_\alpha^\infty \subset F_\alpha^\infty, \quad p < q.$$

We can also define the space f_α^∞ of entire functions $f : \mathbb{C} \rightarrow \mathbb{C}$ such that

$$\lim_{|z| \rightarrow \infty} |f(z)| e^{-\frac{\alpha}{2}|z|^2} = 0.$$

The inclusions between the spaces are as follows:

$$F_\alpha^1 \subset F_\alpha^p \subset F_\alpha^q \subset f_\alpha^\infty \subset F_\alpha^\infty, \quad p < q.$$

The polynomials belong to F_α^p and are dense in F_α^p for $1 \leq p < \infty$.

We can also define the space f_α^∞ of entire functions $f : \mathbb{C} \rightarrow \mathbb{C}$ such that

$$\lim_{|z| \rightarrow \infty} |f(z)| e^{-\frac{\alpha}{2}|z|^2} = 0.$$

The inclusions between the spaces are as follows:

$$F_\alpha^1 \subset F_\alpha^p \subset F_\alpha^q \subset f_\alpha^\infty \subset F_\alpha^\infty, \quad p < q.$$

The polynomials belong to F_α^p and are dense in F_α^p for $1 \leq p < \infty$. For $1 \leq p < \infty$ and $1/p + 1/q = 1$,

$$(F_\alpha^p)^* = F_\alpha^q$$

(in the sense that there is a topological isomorphism).

Vector-Valued Functions

We will suppose that X is a complex Banach space and consider vector-valued functions $f : \mathbb{C} \rightarrow X$.

- We will work with vector-valued holomorphic functions.
- If $f : \mathbb{C} \rightarrow X$ is entire, then there exist unique $x_n \in X$ with $n \geq 0$ such that

$$f(z) = \sum_{n=0}^{\infty} x_n z^n,$$

for every $z \in \mathbb{C}$, where the convergence is absolute and uniform on compact subsets.

- To integrate functions $f : \mathbb{C} \rightarrow X$, we use the Bochner integral.

Definition. The spaces $F_\alpha^p(X)$

For a Banach space X , $1 \leq p < \infty$ and $\alpha > 0$, the Fock space $F_\alpha^p(X)$ consists of entire functions $f: \mathbb{C} \rightarrow X$ such that

$$\|f\|_{p,\alpha,X}^p = \frac{\alpha p}{2\pi} \int_{\mathbb{C}} \|f(z)\|^p e^{-\frac{\alpha p}{2}|z|^2} dz < \infty$$

- $F_\alpha^p(X)$ may be viewed as a subspace of $L^p(d\lambda_{\alpha p/2}, X)$ where

$$d\lambda_{\alpha p/2}(z) = \frac{\alpha p}{2} e^{-\frac{\alpha p}{2}|z|^2} dA(z).$$

For $p = \infty$, $F_\alpha^\infty(X)$ consists of entire functions $f: \mathbb{C} \rightarrow X$ such that

$$\|f\|_{\infty,\alpha,X} = \sup_{z \in \mathbb{C}} \|f(z)\| e^{-\frac{\alpha}{2}|z|^2} < \infty.$$

For $\alpha > 0$, the space $f_\alpha^\infty(X)$ consists of entire functions $f: \mathbb{C} \rightarrow X$ such that

$$\lim_{|z| \rightarrow \infty} \|f(z)\| e^{-\frac{\alpha}{2}|z|^2} = 0.$$

Theorem

The spaces $F_\alpha^p(X)$ and $f_\alpha^\infty(X)$ are Banach spaces. If H is a separable Hilbert space, $F_\alpha^2(H)$ is a separable Hilbert space.

The inclusions between the spaces are as follows:

$$F_\alpha^1(X) \subset F_\alpha^p(X) \subset F_\alpha^q(X) \subset f_\alpha^\infty(X) \subset F_\alpha^\infty(X), \quad p < q.$$

Polynomials with coefficients in X belong to all these spaces. If we define $u_n(z) = z^n$ and, for each $x \in X$,

$$x \otimes u_n(z) = xz^n,$$

with $n \geq 0$, then

$$\|x \otimes u_n\|_{p,\alpha,X}^p = \|x\| \left(\frac{2}{\alpha p} \right)^{\frac{np}{2}} \Gamma \left(\frac{np}{2} + 1 \right) \quad \text{for } 1 \leq p < \infty,$$

$$\|x \otimes u_n\|_{\infty,\alpha,X} = \|x\| \left(\frac{n}{\alpha e} \right)^{\frac{n}{2}}.$$

Weak versions of $F_\alpha^p(X)$

Definition. The spaces $F_\alpha^{w,p}(X)$

We define the space $F_\alpha^{w,p}(X)$ as the set of entire functions $f : \mathbb{C} \rightarrow X$ such that

$$x^* \circ f \in F_\alpha^p \quad \text{for every } x^* \in X^*.$$

$$\|f\|_{w,p,\alpha,X} = \sup_{\|x^*\|_{X^*} \leq 1} \|x^* \circ f\|_{p,\alpha}.$$

$F_\alpha^{w,p}(X)$ can be identified isometrically with a subspace of $B(X^*, F_\alpha^p)$: each f is identified with $T_f : X^* \rightarrow F_\alpha^p$, defined by

$$T_f x^* = x^* \circ f.$$

$F_\alpha^{w,p}(X)$ is a Banach space.

If the Banach space is a dual space, we have another weak version.

Definition. The spaces $F_{\alpha}^{w*,p}(X^*)$

We define the space $F_{\alpha}^{w*,p}(X^*)$ as the set of entire functions $f : \mathbb{C} \rightarrow X^*$ such that

$$f(\cdot)(x) \in F_{\alpha}^p \quad \text{for every } x \in X.$$

$$\|f\|_{w,p,\alpha,X} = \sup_{\|x\| \leq 1} \|f(\cdot)(x)\|_{p,\alpha}.$$

If the Banach space is a dual space, we have another weak version.

Definition. The spaces $F_{\alpha}^{w*,p}(X^*)$

We define the space $F_{\alpha}^{w*,p}(X^*)$ as the set of entire functions $f : \mathbb{C} \rightarrow X^*$ such that

$$f(\cdot)(x) \in F_{\alpha}^p \quad \text{for every } x \in X.$$

$$\|f\|_{w,p,\alpha,X} = \sup_{\|x\| \leq 1} \|f(\cdot)(x)\|_{p,\alpha}.$$

This space is isometrically isomorphic to $B(X, F_{\alpha}^p)$.

Inclusions

$$F_{\alpha}^p(X) \subseteq F_{\alpha}^{w,p}(X)$$

$$F_{\alpha}^{w,p}(X^*) \subseteq F_{\alpha}^{w*,p}(X^*)$$

If $p = \infty$, the spaces also coincide and their norms are equal.

If X is finite-dimensional, $F_{\alpha}^p(X) = F_{\alpha}^{w,p}(X)$ and their norms are equivalent.

If X is reflexive, $F_{\alpha}^{w,p}(X^*) = F_{\alpha}^{w*,p}(X^*)$ and the norms coincide.

When do the spaces $F_{\alpha}^p(X)$ and $F_{\alpha}^{w,p}(X)$ coincide?

As a corollary of the characterization of the boundedness of the composition operators $C_\varphi : F_\alpha^{w,p}(X) \rightarrow F_\alpha^p(X)$ for $1 < p < \infty$, J. Chen, B. He and M. Wang obtained the following result.

Theorem

If X is an infinite-dimensional Banach space,

$$F_\alpha^p(X) \subsetneq F_\alpha^{w,p}(X)$$

- for $2 \leq p < \infty$ (J. Chen, M. Wang, 2021).
- for $1 < p < 2$ (J. Chen, B. He, M. Wang, 2025).

The space $F_\alpha^2(H)$

If H is a separable Hilbert space with Hermitian inner product $(\cdot, \cdot)$, then $F_\alpha^2(H)$ is a Hilbert space with inner product

$$\langle f, g \rangle = \alpha \int_{\mathbb{C}} (f(z), g(z)) e^{-\alpha|z|^2} dA(z), \quad f, g \in F_\alpha^2(H).$$

The space $F_\alpha^2(H)$

If H is a separable Hilbert space with Hermitian inner product $(\cdot, \cdot)$, then $F_\alpha^2(H)$ is a Hilbert space with inner product

$$\langle f, g \rangle = \alpha \int_{\mathbb{C}} (f(z), g(z)) e^{-\alpha|z|^2} dA(z), \quad f, g \in F_\alpha^2(H).$$

Orthonormal basis

An orthonormal basis of $F_\alpha^2(H)$ is

$$\{h_m \otimes e_n\}_{m,n}$$

where $\{h_m\}_{m \in \mathbb{N}}$ is an orthonormal basis of H and

$$e_n = u_n / \|u_n\|_{2,\alpha}.$$

Membership condition for $F_\alpha^2(H)$ and $F_\alpha^{w,2}(H)$

Theorem

Let $f : \mathbb{C} \rightarrow H$ be an entire function whose Taylor expansion is $f(z) = \sum_{n=0}^{\infty} x_n z^n$. Then

$$\bullet f \in F_\alpha^2(H) \iff \sum_{n=0}^{\infty} \frac{n!}{\alpha^n} \|x_n\|^2 < \infty \text{ and } \|f\|_{2,\alpha,H}^2 = \sum_{n=0}^{\infty} \frac{n!}{\alpha^n} \|x_n\|^2.$$

Membership condition for $F_\alpha^2(H)$ and $F_\alpha^{w,2}(H)$

Theorem

Let $f : \mathbb{C} \rightarrow H$ be an entire function whose Taylor expansion is $f(z) = \sum_{n=0}^{\infty} x_n z^n$. Then

- $f \in F_\alpha^2(H) \iff \sum_{n=0}^{\infty} \frac{n!}{\alpha^n} \|x_n\|^2 < \infty$ and $\|f\|_{2,\alpha,H}^2 = \sum_{n=0}^{\infty} \frac{n!}{\alpha^n} \|x_n\|^2$.
- $f \in F_\alpha^{w,2}(H) \iff \sum_{n=0}^{\infty} \frac{n!}{\alpha^n} |(x_n, x)|^2 < \infty$ for all $x \in H$.

$F_\alpha^2(H)$ and $F_\alpha^{w,2}(H)$

When $p = 2$ and H is a separable Hilbert space, we can give an explicit example of a function showing that $F_\alpha^2(H) \subsetneq F_\alpha^{w,2}(H)$:

- It suffices to find a sequence $(x_n) \in H$ such that $\sum_{n=0}^{\infty} \frac{n!}{\alpha^n} \|x_n\|^2 = \infty$ and $\sum_{n=0}^{\infty} \frac{n!}{\alpha^n} |(x_n, x)|^2 < \infty$ for all $x \in H$.

If $(h_n)_{n \geq 0}$ is an orthonormal basis of H , we can take $x_n = \sqrt{\frac{\alpha^n}{n!}} h_n$.

- $\sum_{n=0}^{\infty} \frac{n!}{\alpha^n} \|x_n\|^2 = \infty$,
- $\sum_{n=0}^{\infty} \frac{n!}{\alpha^n} |(x_n, x)|^2 = \sum_{n=0}^{\infty} |(x, h_n)|^2 = \|x\|^2$.

(O. Blasco, M. Ventura, 2026)

Theorem

$F_\alpha^{w,2}(H)$ is isometrically isomorphic to $B(H, F_\alpha^2)$.

Theorem

$F_\alpha^{w,2}(H)$ is isometrically isomorphic to $B(H, F_\alpha^2)$.

What kind of operators are $F_\alpha^2(H)$?

Hilbert–Schmidt Operators

Definition. Hilbert–Schmidt operators

Let H_1 and H_2 be separable Hilbert spaces. An operator $T \in B(H_1, H_2)$ is Hilbert–Schmidt if, for every orthonormal basis $\{x_n\}_{n \in \mathbb{N}}$ of H_1 , we have

$$\|T\|_{HS}^2 := \sum_{n=1}^{\infty} \|Tx_n\|^2 < \infty.$$

$\|\cdot\|_{HS}$ does not depend on the orthonormal basis and defines a norm.

Theorem (O. Blasco, M. Ventura, 2026)

$F_{\alpha}^2(H)$ is isometrically isomorphic to $B_{HS}(H, F_{\alpha}^2)$, identifying f with T_f .

Density of polynomials in $F_\alpha^p(X)$

Theorem

Let $1 \leq p < \infty$. The set of polynomials with coefficients in X is dense in $F_\alpha^p(X)$. These polynomials are also dense in $f_\alpha^\infty(X)$.

- The proof is based on the fact that if $f \in F_\alpha^p(X)$, $1 \leq p < \infty$, or $f \in f_\alpha^\infty(X)$, then f can be approximated by $f_r(z) = f(rz)$, $0 < r < 1$.
- The Taylor series of f_r converges absolutely in the norm of $F_\alpha^p(X)$ and in $F_\alpha^\infty(X)$, respectively.

Density of polynomials in $F_\alpha^p(X)$

Theorem

Let $1 \leq p < \infty$. The set of polynomials with coefficients in X is dense in $F_\alpha^p(X)$. These polynomials are also dense in $f_\alpha^\infty(X)$.

- The proof is based on the fact that if $f \in F_\alpha^p(X)$, $1 \leq p < \infty$, or $f \in f_\alpha^\infty(X)$, then f can be approximated by $f_r(z) = f(rz)$, $0 < r < 1$.
- The Taylor series of f_r converges absolutely in the norm of $F_\alpha^p(X)$ and in $F_\alpha^\infty(X)$, respectively.

Are the polynomials also dense in $F_\alpha^{w,p}(X)$?

Projection Operator

For $1 \leq p < \infty$, for each $f \in F_\alpha^p(X)$

$$f(z) = \int_{\mathbb{C}} f(w) e^{\alpha z \bar{w}} d\lambda_\alpha(w)$$

Definition of $P_{\alpha,X}$

We define the projection operator $P_{\alpha,X} : L^p(d\lambda_{\alpha p/2}, X) \rightarrow F_\alpha^p(X)$ by

$$P_{\alpha,X} f(z) = \int_{\mathbb{C}} f(w) e^{\alpha z \bar{w}} d\lambda_\alpha(w).$$

Theorem

The operator $P_{\alpha,X}$ is well defined and bounded. Moreover,

$$\|P_{\alpha,X}\| \leq 2.$$

Duality in other spaces of vector-valued functions

Let $1 < p < \infty$, $1/p + 1/q = 1$.

$$L^p(\mu, X)^* = L^q(\mu, X^*) \iff X^* \text{ has the Radon-Nikodym property}$$

$$H^p(X)^* = H^q(X^*) \iff X \text{ has the UMD property}$$

$$B_p(X)^* = B_q(X^*) \text{ without additional hypotheses}$$

Theorem

If $1 < p < \infty$ and $1/p + 1/q = 1$, then the following duality results hold:

$$F_{\alpha}^p(X)^* = F_{\alpha}^q(X^*)$$

$$F_{\alpha}^1(X)^* = F_{\alpha}^{\infty}(X^*)$$

These identities are understood as topological isomorphisms, not necessarily as isometries. When X is reflexive, we also have

$$f_{\alpha}^{\infty}(X)^* = F_{\alpha}^1(X^*).$$

Sketch of the proof

The isomorphism is defined as follows:

$$\Gamma : F_{\alpha}^q(X^*) \rightarrow F_{\alpha}^p(X)^*$$

$$\Gamma g(f) = \alpha \int_{\mathbb{C}} g(z)(f(\bar{z}))e^{-\alpha|z|^2}dA(z).$$

We may also consider

$$\Gamma : F_{\alpha}^{\infty}(X^*) \rightarrow F_{\alpha}^1(X)^*$$

and

$$\Gamma : F_{\alpha}^1(X^*) \rightarrow f_{\alpha}^{\infty}(X)^*.$$

- Hölder's inequality shows that Γg is well defined and

$$\|\Gamma g\|_{F_{\alpha}^p(X)^*} \leq 2p^{-\frac{1}{p}}q^{-\frac{1}{q}}\|g\|_{q,\alpha,X^*},$$

allowing p to be 1 or ∞ .

Therefore, Γ is linear and continuous in every case.

If

$$g(z) = \sum_{n=0}^{\infty} x_n^* \cdot z^n,$$

then

$$\Gamma g(x \otimes u_n) = \frac{n!}{\alpha^n} x_n^*(x).$$

Therefore, for surjectivity, if $F \in F_{\alpha}^p(X)^*$ (or in the corresponding dual space), we consider

$$g(z) = \sum_{n=0}^{\infty} x_n^* \cdot z^n,$$

where $x_n^* \in X^*$ is defined by

$$x_n^*(x) = \frac{\alpha^n}{n!} F(x \otimes u_n).$$

Using the previous identity, if g belongs to $F_\alpha^q(X^*)$ (or in the corresponding space), then

$$Tg(x \otimes u_n) = \frac{n!}{\alpha^n} x_n^*(x) = F(x \otimes u_n).$$

By linearity, the equality holds for polynomials; by density, it holds for every function in $F_\alpha^p(X)$.



$$F \in F_\alpha^1(X)^* \implies g \in F_\alpha^\infty(X).$$

Compute $\|g\|_{\infty, \alpha, X^*}$ directly.



$$F \in F_\alpha^p(X)^* \implies g \in F_\alpha^q(X).$$

Show that

$$\phi(f) := \alpha \int_{\mathbb{C}} g(z)(f(z))e^{-\alpha|z|^2} dz$$

defines a linear and continuous functional

$$\phi : L^p(dA, X) \rightarrow \mathbb{C}.$$

The boundedness of $P_{\alpha, X}$ implies the continuity of ϕ .



$$F \in f_{\alpha}^{\infty}(X)^* \implies g \in F_{\alpha}^1(X),$$

and a similar argument, which requires X to be reflexive plus the previously proved result $F_{\alpha}^1(X)^* = F_{\alpha}^{\infty}(X^*)$.

No additional geometric property of the Banach space X is needed.

Thank you for your attention!

-  J. Chen B. He, M. Wang (2025). *Absolutely summing Carleson embeddings on weighted Fock spaces with A_∞ -type weights*. arXiv: 2505.16109 [math.FA]. URL: <https://arxiv.org/abs/2505.16109>.
-  K. Zhu (2012). *Analysis on Fock Spaces*. Graduate Texts in Mathematics. Springer US. ISBN: 9781441988010.
-  O. Blasco, M. Ventura (2026). “Some results on vector-valued Fock spaces”. In: *Journal of Mathematical Sciences*.

Matrix-valued Bispectrality: from the matrix Bochner problem to matrix exceptional polynomials

Ignacio Bono Parisi

EARCO 2026, Encuentro de Análisis Real y Complejo

Universidad Nacional de Córdoba, Argentina

Mayo, 2026

Definición: un peso matricial W de tamaño N con soporte en (a, b) es una función

$$W : \mathbb{R} \rightarrow M_N(\mathbb{C})$$

tal que

- $W(x)$ es Hermitiana definida positiva a.e. en (a, b) ,
- $W(x) = 0$ fuera de (a, b) ,
- $\int_{\mathbb{R}} x^n W(x) dx < \infty$ para todo $n \geq 0$.

Definición: un peso matricial W define un producto interno en el álgebra de polinomios matriciales $M_N(\mathbb{C})[x]$,

$$\langle P, Q \rangle_W = \int_{\mathbb{R}} P(x)W(x)Q(x)^* dx,$$

para todo $P, Q \in M_N(\mathbb{C})[x]$.

Definición: un peso matricial W define un producto interno en el álgebra de polinomios matriciales $M_N(\mathbb{C})[x]$,

$$\langle P, Q \rangle_W = \int_{\mathbb{R}} P(x)W(x)Q(x)^* dx,$$

para todo $P, Q \in M_N(\mathbb{C})[x]$.

Con este producto interno se construye una única sucesión de polinomios mónicos ortogonales $\{P_n(x)\}_{n \geq 0}$.

Sucesión de polinomios mónicos ortogonales:

- $\deg(P_n(x)) = n$,
- $P_n(x) = x^n I + x^{n-1} T_{n-1} + \cdots + T_0$, $T_j \in M_N(\mathbb{C})$,
- $\langle P_n, P_m \rangle_W = 0$ para todo $n \neq m$.

Sucesión de polinomios mónicos ortogonales:

- $\deg(P_n(x)) = n$,
- $P_n(x) = x^n I + x^{n-1} T_{n-1} + \cdots + T_0$, $T_j \in M_N(\mathbb{C})$,
- $\langle P_n, P_m \rangle_W = 0$ para todo $n \neq m$.

Forman una base como $M_N(\mathbb{C})$ -módulo de $M_N(\mathbb{C})[x]$.

Sucesión de polinomios mónicos ortogonales:

- $\deg(P_n(x)) = n$,
- $P_n(x) = x^n I + x^{n-1} T_{n-1} + \cdots + T_0$, $T_j \in M_N(\mathbb{C})$,
- $\langle P_n, P_m \rangle_W = 0$ para todo $n \neq m$.

Forman una base como $M_N(\mathbb{C})$ -módulo de $M_N(\mathbb{C})[x]$.

Siempre satisfacen una relación de recurrencia de tres términos

$$P_n(x)x = P_{n+1}(x) + B_n P_n(x) + C_n P_{n-1}(x)$$

Sucesión de polinomios mónicos ortogonales:

- $\deg(P_n(x)) = n$,
- $P_n(x) = x^n I + x^{n-1} T_{n-1} + \cdots + T_0$, $T_j \in M_N(\mathbb{C})$,
- $\langle P_n, P_m \rangle_W = 0$ para todo $n \neq m$.

Forman una base como $M_N(\mathbb{C})$ -módulo de $M_N(\mathbb{C})[x]$.

Siempre satisfacen una relación de recurrencia de tres términos

$$P_n(x)x = P_{n+1}(x) + B_n P_n(x) + C_n P_{n-1}(x)$$

$$(\mathcal{S} + B_n + C_n \mathcal{S}^{-1}) \cdot P_n(x) = P_n(x)x$$

.

Consideramos operadores diferenciales de la forma

$$D = \sum_{j=0}^m \partial^j F_j(x),$$

donde $\partial = \frac{d}{dx}$ y $F_j(x) \in M_N(\mathbb{C})[x]$.

Consideramos operadores diferenciales de la forma

$$D = \sum_{j=0}^m \partial^j F_j(x),$$

donde $\partial = \frac{d}{dx}$ y $F_j(x) \in M_N(\mathbb{C})[x]$.

Estos operadores actúan a derecha con la acción dada por

$$Q(x) \cdot D = \sum_{j=0}^m \partial^j (Q(x)) F_j(x).$$

Problema de Bochner

Dado un peso matricial W , decimos que su sucesión de polinomios ortogonales $P_n(x)$ es autofunción de un operador diferencial D si se cumple que

$$P_n(x) \cdot D = \Lambda_n(D)P_n(x), \quad \text{para todo } n \geq 0,$$

con $\Lambda_n(D) \in M_N(\mathbb{C})$.

Problema de Bochner

Dado un peso matricial W , decimos que su sucesión de polinomios ortogonales $P_n(x)$ es autofunción de un operador diferencial D si se cumple que

$$P_n(x) \cdot D = \Lambda_n(D)P_n(x), \quad \text{para todo } n \geq 0,$$

con $\Lambda_n(D) \in M_N(\mathbb{C})$.

Cuando esto ocurre, el operador diferencial D junto con el operador discreto que define la relación de recurrencia de tres términos $\mathcal{L} = \mathcal{S} + B_n + C_n\mathcal{S}^{-1}$, se tiene que

$$P_n(x) \cdot D = \Lambda_n(D)P_n(x),$$

$$\mathcal{L} \cdot P_n(x) = P_n(x)x.$$

Problema de Bochner

Dado un peso matricial W , decimos que su sucesión de polinomios ortogonales $P_n(x)$ es autofunción de un operador diferencial D si se cumple que

$$P_n(x) \cdot D = \Lambda_n(D)P_n(x), \quad \text{para todo } n \geq 0,$$

con $\Lambda_n(D) \in M_N(\mathbb{C})$.

Cuando esto ocurre, el operador diferencial D junto con el operador discreto que define la relación de recurrencia de tres términos $\mathcal{L} = \mathcal{S} + B_n + C_n\mathcal{S}^{-1}$, se tiene que

$$\begin{aligned} P_n(x) \cdot D &= \Lambda_n(D)P_n(x), \\ \mathcal{L} \cdot P_n(x) &= P_n(x)x. \end{aligned}$$

Decimos en este caso que la sucesión $P_n(x)$ es biespectral.

Problema de Bochner

Nos van a interesar en particular los pesos cuya sucesión sea autofunción de algún operador de segundo orden

$$P_n(x) \cdot D = P_n''(x)F_2(x) + P_n'(x)F_1(x) + P_n(x)F_0(x) = \Lambda_n(D)P_n(x).$$

Problema de Bochner

Nos van a interesar en particular los pesos cuya sucesión sea autofunción de algún operador de segundo orden

$$P_n(x) \cdot D = P_n''(x)F_2(x) + P_n'(x)F_1(x) + P_n(x)F_0(x) = \Lambda_n(D)P_n(x).$$

El Problema Matricial de Bochner

Clasificar todos los pesos matriciales W de tamaño $N \times N$ tales que su sucesión de polinomios ortogonales $P_n(x)$ sea autofunción de algún operador diferencial de segundo orden.

Problema de Bochner

Nos van a interesar en particular los pesos cuya sucesión sea autofunción de algún operador de segundo orden

$$P_n(x) \cdot D = P_n''(x)F_2(x) + P_n'(x)F_1(x) + P_n(x)F_0(x) = \Lambda_n(D)P_n(x).$$

El Problema Matricial de Bochner

Clasificar todos los pesos matriciales W de tamaño $N \times N$ tales que su sucesión de polinomios ortogonales $P_n(x)$ sea autofunción de algún operador diferencial de segundo orden.

El problema fue originalmente planteado y resuelto por S. Bochner en 1929 para el caso escalar.

Problema de Bochner

Nos van a interesar en particular los pesos cuya sucesión sea autofunción de algún operador de segundo orden

$$P_n(x) \cdot D = P_n''(x)F_2(x) + P_n'(x)F_1(x) + P_n(x)F_0(x) = \Lambda_n(D)P_n(x).$$

El Problema Matricial de Bochner

Clasificar todos los pesos matriciales W de tamaño $N \times N$ tales que su sucesión de polinomios ortogonales $P_n(x)$ sea autofunción de algún operador diferencial de segundo orden.

El problema fue originalmente planteado y resuelto por S. Bochner en 1929 para el caso escalar.

Luego, en 1997, A. Durán propuso la extensión de este problema al caso matricial.

Caso escalar ($N = 1$): las únicas soluciones, salvo un cambio de variables afín, son las tres familias de pesos escalares clásicos

Caso escalar ($N = 1$): las únicas soluciones, salvo un cambio de variables afín, son las tres familias de pesos escalares clásicos

- **Hermite :**

$$w(x) = e^{-x^2}, \quad \delta = \partial^2 + \partial(-2x).$$

- **Laguerre :**

$$w(x) = x^\alpha e^{-x}, \quad \delta_\alpha = \partial^2 x + \partial(\alpha + 1 - x).$$

- **Jacobi :**

$$w(x) = (1-x)^\alpha (1+x)^\beta$$
$$\delta_{\alpha,\beta} = \partial^2(1-x^2) + \partial(\beta - \alpha - (\alpha + \beta + 2)x).$$

Los respectivos polinomios ortogonales cumplen

- Hermite:

$$h_n(x) \cdot \delta = -2nh_n(x),$$

- Laguerre:

$$\ell_n^{(\alpha)}(x) \cdot \delta_\alpha = -n\ell_n^{(\alpha)}(x),$$

- Jacobi:

$$p_n^{(\alpha,\beta)}(x) \cdot \delta_{\alpha,\beta} = -n(n + \alpha + \beta + 1)p_n^{(\alpha,\beta)}(x).$$

Caso matricial ($N > 1$):

Caso matricial ($N > 1$):

Al principio no se encontraban ejemplos nuevos no triviales.

Caso matricial ($N > 1$):

Al principio no se encontraban ejemplos nuevos no triviales.

El primer ejemplo no trivial fue encontrado alrededor del 2001 por Pacharoni, Tirao y Grünbaum a través de la teoría de representaciones.

Caso matricial ($N > 1$):

Al principio no se encontraban ejemplos nuevos no triviales.

El primer ejemplo no trivial fue encontrado alrededor del 2001 por Pacharoni, Tirao y Grünbaum a través de la teoría de representaciones.

A partir de ahí la teoría cobra sus fuerzas y numerosos nuevos ejemplos matriciales son encontrados utilizando diferentes técnicas.

Caso matricial ($N > 1$):

Al principio no se encontraban ejemplos nuevos no triviales.

El primer ejemplo no trivial fue encontrado alrededor del 2001 por Pacharoni, Tirao y Grünbaum a través de la teoría de representaciones.

A partir de ahí la teoría cobra sus fuerzas y numerosos nuevos ejemplos matriciales son encontrados utilizando diferentes técnicas.

Sin embargo, el problema de clasificar todas las soluciones seguía estando lejos de resolverse.

Problema de Bochner

Un paso en la dirección correcta para la clasificación del problema matricial de Bochner fue comenzar a estudiar el álgebra $\mathcal{D}(W)$.

Problema de Bochner

Un paso en la dirección correcta para la clasificación del problema matricial de Bochner fue comenzar a estudiar el álgebra $\mathcal{D}(W)$.

Definición: sea W un peso matricial, se define el álgebra $\mathcal{D}(W)$ como

$$\mathcal{D}(W) = \left\{ D = \sum_{j=0}^m \partial^j F_j(x) : P_n(x) \cdot D = \Lambda_n(D) P_n(x) \right\}.$$

Problema de Bochner

Un paso en la dirección correcta para la clasificación del problema matricial de Bochner fue comenzar a estudiar el álgebra $\mathcal{D}(W)$.

Definición: sea W un peso matricial, se define el álgebra $\mathcal{D}(W)$ como

$$\mathcal{D}(W) = \left\{ D = \sum_{j=0}^m \partial^j F_j(x) : P_n(x) \cdot D = \Lambda_n(D) P_n(x) \right\}.$$

Definición: dado un peso W de tamaño N , decimos que $\mathcal{D}(W)$ es *completa* si existen N operadores no nulos $D_1, \dots, D_N$ tales que

$$D_i D_j = 0 \quad \text{para todo } i \neq j$$

y $D_1 + \dots + D_N$ está en el centro de $\mathcal{D}(W)$.

Definición: Decimos que un peso W es una *transformación de Darboux* de otro peso $\widetilde{W}$, si existe una factorización

$$D = D_1 D_2 \in \mathcal{D}(W)$$

tal que

$$P_n(x) \cdot D_1 = A_n \widetilde{P}_n(x).$$

Definición: Decimos que un peso W es una *transformación de Darboux* de otro peso $\widetilde{W}$, si existe una factorización

$$D = D_1 D_2 \in \mathcal{D}(W)$$

tal que

$$P_n(x) \cdot D_1 = A_n \widetilde{P}_n(x).$$

En ese caso, se cumple que al conmutar los factores,

$$\widetilde{D} = D_2 D_1 \text{ está en } \mathcal{D}(\widetilde{W}).$$

Problema de Bochner

En 2022, bajo una hipótesis extra, Casper y Yakimov dan el primer teorema de clasificación para el caso matricial del problema de Bochner.

Problema de Bochner

En 2022, bajo una hipótesis extra, Casper y Yakimov dan el primer teorema de clasificación para el caso matricial del problema de Bochner.

Teorema de Clasificación (R. Casper y M. Yakimov)

Sea W un peso $N \times N$ solución al problema matricial de Bochner,

Problema de Bochner

En 2022, bajo una hipótesis extra, Casper y Yakimov dan el primer teorema de clasificación para el caso matricial del problema de Bochner.

Teorema de Clasificación (R. Casper y M. Yakimov)

Sea W un peso $N \times N$ solución al problema matricial de Bochner, entonces $\mathcal{D}(W)$ es *completa* si y solo si W es una *transformación de Darboux* de un peso diagonal

$$\widetilde{W}(x) = w_1(x) \oplus \cdots \oplus w_N(x) = \begin{pmatrix} w_1(x) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & w_N(x) \end{pmatrix},$$

donde $w_1, w_2, \dots, w_N$ son pesos escalares clásicos de Hermite, Laguerre o Jacobi.

Problema de Bochner

Ejemplo 2×2 :

Problema de Bochner

Ejemplo 2×2 : el peso matricial

$$W(x) = e^{-x^2} \begin{pmatrix} 1 + a^2 x^2 & ax \\ ax & 1 \end{pmatrix}$$

tiene un operador de orden 2 en su álgebra,

Ejemplo 2×2 : el peso matricial

$$W(x) = e^{-x^2} \begin{pmatrix} 1 + a^2 x^2 & ax \\ ax & 1 \end{pmatrix}$$

tiene un operador de orden 2 en su álgebra,

$$\partial^2 I + \partial \begin{pmatrix} -2x & 2a \\ 0 & -2x \end{pmatrix} - \begin{pmatrix} \frac{4}{a^2} + 2 & 0 \\ 0 & \frac{4}{a^2} \end{pmatrix} \in \mathcal{D}(W)$$

Problema de Bochner

Ejemplo 2×2 : el peso matricial

$$W(x) = e^{-x^2} \begin{pmatrix} 1 + a^2 x^2 & ax \\ ax & 1 \end{pmatrix}$$

tiene un operador de orden 2 en su álgebra,

$$\partial^2 I + \partial \begin{pmatrix} -2x & 2a \\ 0 & -2x \end{pmatrix} - \begin{pmatrix} \frac{4}{a^2} + 2 & 0 \\ 0 & \frac{4}{a^2} \end{pmatrix} \in \mathcal{D}(W)$$

y proviene de una transformación de Darboux del peso diagonal de Hermite

$$\widetilde{W}(x) = \begin{pmatrix} e^{-x^2} & 0 \\ 0 & e^{-x^2} \end{pmatrix}.$$

Esta hipótesis extra de que el álgebra $\mathcal{D}(W)$ sea completa era lo suficientemente natural como para englobar prácticamente todos los ejemplos de la literatura.

Esta hipótesis extra de que el álgebra $\mathcal{D}(W)$ sea completa era lo suficientemente natural como para englobar prácticamente todos los ejemplos de la literatura.

Hasta que encontramos que algunas soluciones al problema matricial de Bochner no cumplen con esa condición y se escapan de esta clasificación.

Ejemplos singulares al problema de Bochner

El peso matricial 2×2 del tipo Hermite dado por

$$W(x) = e^{-x^2} \begin{pmatrix} e^{2bx} + a^2x^2 & ax \\ ax & 1 \end{pmatrix}, \quad a, b, \in \mathbb{R} - \{0\},$$

Ejemplos singulares al problema de Bochner

El peso matricial 2×2 del tipo Hermite dado por

$$W(x) = e^{-x^2} \begin{pmatrix} e^{2bx} + a^2x^2 & ax \\ ax & 1 \end{pmatrix}, \quad a, b, \in \mathbb{R} - \{0\},$$

Ejemplos singulares al problema de Bochner

El peso matricial 2×2 del tipo Hermite dado por

$$W(x) = e^{-x^2} \begin{pmatrix} e^{2bx} + a^2 x^2 & ax \\ ax & 1 \end{pmatrix}, \quad a, b, \in \mathbb{R} - \{0\},$$

cumple que el operador

$$D = \partial^2 I + \partial \begin{pmatrix} -2x + 2b & -2abx + 2a \\ 0 & -2x \end{pmatrix} + \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}$$

está en $\mathcal{D}(W)$, luego es solución al problema matricial de Bochner.

Ejemplos singulares al problema de Bochner

El peso matricial 2×2 del tipo Hermite dado por

$$W(x) = e^{-x^2} \begin{pmatrix} e^{2bx} + a^2 x^2 & ax \\ ax & 1 \end{pmatrix}, \quad a, b, \in \mathbb{R} - \{0\},$$

cumple que el operador

$$D = \partial^2 I + \partial \begin{pmatrix} -2x + 2b & -2abx + 2a \\ 0 & -2x \end{pmatrix} + \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}$$

está en $\mathcal{D}(W)$, luego es solución al problema matricial de Bochner.

Calculando operadores en $\mathcal{D}(W)$ de distintos órdenes obteníamos que todos eran combinaciones de potencias de D .

Ejemplos singulares al problema de Bochner

Finalmente, logramos demostrar el siguiente teorema sobre la estructura de $\mathcal{D}(W)$

Ejemplos singulares al problema de Bochner

Finalmente, logramos demostrar el siguiente teorema sobre la estructura de $\mathcal{D}(W)$

Teorema (I. B. P. - I. Pacharoni)

El álgebra $\mathcal{D}(W)$ es un álgebra polinomial sobre el operador de segundo orden D , es decir, $\mathcal{D}(W) = \mathbb{C}[D]$.

Ejemplos singulares al problema de Bochner

Finalmente, logramos demostrar el siguiente teorema sobre la estructura de $\mathcal{D}(W)$

Teorema (I. B. P. - I. Pacharoni)

El álgebra $\mathcal{D}(W)$ es un álgebra polinomial sobre el operador de segundo orden D , es decir, $\mathcal{D}(W) = \mathbb{C}[D]$.

Como consecuencia directa, obtenemos el siguiente corolario.

Corolario (I. B. P. - I. Pacharoni)

El álgebra $\mathcal{D}(W)$ no es completa, luego W no es una transformación de Darboux de ninguna diagonal de pesos escalares clásicos.

Ejemplos singulares al problema de Bochner

Finalmente, logramos demostrar el siguiente teorema sobre la estructura de $\mathcal{D}(W)$

Teorema (I. B. P. - I. Pacharoni)

El álgebra $\mathcal{D}(W)$ es un álgebra polinomial sobre el operador de segundo orden D , es decir, $\mathcal{D}(W) = \mathbb{C}[D]$.

Como consecuencia directa, obtenemos el siguiente corolario.

Corolario (I. B. P. - I. Pacharoni)

El álgebra $\mathcal{D}(W)$ no es completa, luego W no es una transformación de Darboux de ninguna diagonal de pesos escalares clásicos.

Llamamos a este peso **Ejemplo singular del problema matricial de Bochner**.

Ejemplos singulares al problema de Bochner

Inmediatamente, nos dimos cuenta que podíamos generalizar este ejemplo a tamaños más grandes.

Ejemplos singulares al problema de Bochner

Inmediatamente, nos dimos cuenta que podíamos generalizar este ejemplo a tamaños más grandes. Y podíamos hacerlo con cualquier familia (Hermite, Laguerre o Jacobi).

Ejemplos singulares al problema de Bochner

Inmediatamente, nos dimos cuenta que podíamos generalizar este ejemplo a tamaños más grandes. Y podíamos hacerlo con cualquier familia (Hermite, Laguerre o Jacobi).

Tomamos la matriz nilpotente

$$A = \begin{pmatrix} 0 & a_1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & a_2 & 0 & a_3 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & a_4 & 0 & a_5 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}_{N \times N}, \quad a_i \in \mathbb{R} - \{0\}.$$

Ejemplos singulares al problema de Bochner

Consideramos los pesos diagonales contruidos con pesos escalares $w_1, \dots, w_N$ del mismo tipo tales que w_i/w_j no es una función racional.

Ejemplos singulares al problema de Bochner

Consideramos los pesos diagonales contruidos con pesos escalares $w_1, \dots, w_N$ del mismo tipo tales que w_i/w_j no es una función racional.

Explícitamente, si $w_b(x) = e^{-x^2+2bx}$, $w_\alpha(x) = e^{-x}x^\alpha$, y $w_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$, tenemos

① *Hermite*:

$$\widetilde{W}_H(x) = w_{b_1}(x) \oplus \cdots \oplus w_{b_N}(x), \text{ con } b_i \neq b_j.$$

② *Laguerre*:

$$\widetilde{W}_L(x) = w_{\alpha_1}(x) \oplus \cdots \oplus w_{\alpha_N}(x), \text{ con } \alpha_i - \alpha_j \notin \mathbb{Z}.$$

③ *Jacobi*:

$$\widetilde{W}_J(x) = w_{\alpha_1,\beta_1}(x) \oplus \cdots \oplus w_{\alpha_N,\beta_N}(x), \text{ con } \alpha_i - \alpha_j \notin \mathbb{Z} \text{ o } \beta_i - \beta_j \notin \mathbb{Z} \text{ y } \alpha_1 + \beta_1 = \alpha_j + \beta_j + 1 + (-1)^j.$$

Ejemplos singulares al problema de Bochner

Construimos los pesos matriciales de tamaño $N \times N$

- Hermite:

$$W_H(x) = T(x) \widetilde{W}_H(x) T(x)^*,$$

- Laguerre:

$$W_L(x) = T(x) \widetilde{W}_L(x) T(x)^*,$$

- Jacobi:

$$W_J(x) = T(x) \widetilde{W}_J(x) T(x)^*$$

donde $T(x) = e^{Ax} = I + Ax$.

Ejemplos singulares al problema de Bochner

Para estos pesos obtuvimos que

Ejemplos singulares al problema de Bochner

Para estos pesos obtuvimos que

Teorema (I. B. P. - I. Pacharoni)

Sea $W = W_H, W_L$, o W_J , entonces existe

$$D = \partial^2 \rho(x)I + \partial F_1(x) + F_0 \in \mathcal{D}(W),$$

donde $\rho(x) = 1, x$, o $1 - x^2$ dependiendo si el peso es de Hermite, Laguerre, o Jacobi, respectivamente.

Ejemplos singulares al problema de Bochner

Para estos pesos obtuvimos que

Teorema (I. B. P. - I. Pacharoni)

Sea $W = W_H, W_L$, o W_J , entonces existe

$$D = \partial^2 \rho(x)I + \partial F_1(x) + F_0 \in \mathcal{D}(W),$$

donde $\rho(x) = 1, x$, o $1 - x^2$ dependiendo si el peso es de Hermite, Laguerre, o Jacobi, respectivamente.

Es decir, todos son solución al problema matricial de Bochner.

Ejemplos singulares al problema de Bochner

También probamos lo siguiente

Ejemplos singulares al problema de Bochner

También probamos lo siguiente

Teorema (I. B. P. - I. Pacharoni)

Sea $W = W_H, W_L$, o W_J . Entonces, el álgebra $\mathcal{D}(W)$ es un álgebra polinomial sobre D , es decir, $\mathcal{D}(W) = \mathbb{C}[D]$.

Ejemplos singulares al problema de Bochner

También probamos lo siguiente

Teorema (I. B. P. - I. Pacharoni)

Sea $W = W_H, W_L$, o W_J . Entonces, el álgebra $\mathcal{D}(W)$ es un álgebra polinomial sobre D , es decir, $\mathcal{D}(W) = \mathbb{C}[D]$.

Como consecuencia, al igual que en el ejemplo 2×2 , concluimos lo siguiente.

Corolario (I. B. P. - I. Pacharoni)

El álgebra $\mathcal{D}(W)$ no es completa, luego $W = W_H, W_L$, o W_J , no es una transformación de Darboux de ninguna diagonal de pesos escalares clásicos.

Transformación de Darboux

Volvamos a la transformación de Darboux.

Transformación de Darboux

Volvamos a la transformación de Darboux.

Consideramos la relación entre pesos matriciales

$W \sim \widetilde{W}$ si y solo si $\widetilde{W}$ es una transformación de Darboux de W .

Volvamos a la transformación de Darboux.

Consideramos la relación entre pesos matriciales

$W \sim \widetilde{W}$ si y solo si $\widetilde{W}$ es una transformación de Darboux de W .

Esta relación define una relación de equivalencias entre pesos matriciales con soporte en el mismo intervalo.

Transformación de Darboux

Clases de equivalencia para los pesos escalares clásicos:

Transformación de Darboux

Clases de equivalencia para los pesos escalares clásicos:

- **(shifted) Hermite :**

$$w_b(x) = e^{-x^2+2bx}, \quad x \in \mathbb{R}, b \in \mathbb{R}.$$

$$w_b \sim w_a \text{ si y sólo si } b = a.$$

Transformación de Darboux

Clases de equivalencia para los pesos escalares clásicos:

- **(shifted) Hermite :**

$$w_b(x) = e^{-x^2+2bx}, \quad x \in \mathbb{R}, b \in \mathbb{R}.$$

$$w_b \sim w_a \text{ si y sólo si } b = a.$$

- **Laguerre :**

$$w_\alpha(x) = x^\alpha e^{-x}, \quad x \in (0, \infty), \quad \alpha > -1.$$

$$w_\alpha \sim w_\beta \text{ si y sólo si } \alpha - \beta \in \mathbb{Z}$$

Transformación de Darboux

Clases de equivalencia para los pesos escalares clásicos:

- **(shifted) Hermite :**

$$w_b(x) = e^{-x^2+2bx}, \quad x \in \mathbb{R}, b \in \mathbb{R}.$$

$$w_b \sim w_a \text{ si y sólo si } b = a.$$

- **Laguerre :**

$$w_\alpha(x) = x^\alpha e^{-x}, \quad x \in (0, \infty), \quad \alpha > -1.$$

$$w_\alpha \sim w_\beta \text{ si y sólo si } \alpha - \beta \in \mathbb{Z}$$

- **Jacobi :**

$$w_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta, \quad x \in (-1, 1), \quad \alpha, \beta > -1.$$

$$w_{\alpha,\beta} \sim w_{\tilde{\alpha},\tilde{\beta}} \text{ si y sólo si } \alpha + \beta = \tilde{\alpha} + \tilde{\beta} \text{ y } \alpha - \tilde{\alpha} \in \mathbb{Z}.$$

Transformación de Darboux

La transformación de Darboux también nos permite relacionar las álgebras de los pesos.

Transformación de Darboux

La transformación de Darboux también nos permite relacionar las álgebras de los pesos.

Teorema (I. B. P - I. Pacharoni)

Si $\widetilde{W}$ es una transformación de Darboux de W , con $D = D_1 D_2$ como en la definición, entonces

$$D_1 \mathcal{D}(\widetilde{W}) D_2 \subset \mathcal{D}(W) \quad \text{y} \quad D_2 \mathcal{D}(W) D_1 \subset \mathcal{D}(\widetilde{W}).$$

Transformación de Darboux

La transformación de Darboux también nos permite relacionar las álgebras de los pesos.

Teorema (I. B. P - I. Pacharoni)

Si $\widetilde{W}$ es una transformación de Darboux de W , con $D = D_1 D_2$ como en la definición, entonces

$$D_1 \mathcal{D}(\widetilde{W}) D_2 \subset \mathcal{D}(W) \quad \text{y} \quad D_2 \mathcal{D}(W) D_1 \subset \mathcal{D}(\widetilde{W}).$$

Más aún, tenemos que $\mathcal{D}(W)$ es conmutativa si y solo si $\mathcal{D}(\widetilde{W})$ es conmutativa.

Transformación de Darboux

Ejemplo:

$$W(x) = e^{-x^2} \begin{pmatrix} 1 + a^2 x^2 & ax \\ ax & 1 \end{pmatrix} \sim \widetilde{W}(x) = \begin{pmatrix} e^{-x^2} & 0 \\ 0 & e^{-x^2} \end{pmatrix},$$

con $D = D_1 D_2 \in \mathcal{D}(W)$.

Transformación de Darboux

Ejemplo:

$$W(x) = e^{-x^2} \begin{pmatrix} 1 + a^2 x^2 & ax \\ ax & 1 \end{pmatrix} \sim \widetilde{W}(x) = \begin{pmatrix} e^{-x^2} & 0 \\ 0 & e^{-x^2} \end{pmatrix},$$

con $D = D_1 D_2 \in \mathcal{D}(W)$. Luego $\mathcal{D}(W)$ contiene los siguientes operadores de orden 2

$$E_1 = D_2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} D_1, \quad E_2 = D_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} D_1,$$

$$E_3 = D_2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} D_1, \quad E_4 = D_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} D_1.$$

Transformación de Darboux

Teorema: el álgebra $\mathcal{D}(W)$ está generada por $\{I, E_1, E_2, E_3, E_4\}$.

$$E_1 = \partial^2 \begin{pmatrix} 0 & -ax \\ 0 & -1 \end{pmatrix} + \partial \begin{pmatrix} 2x & -2a - \frac{2}{a} \\ \frac{2}{a} & 0 \end{pmatrix} + \begin{pmatrix} 2 + \frac{4}{a^2} & 0 \\ 0 & 0 \end{pmatrix},$$

$$E_2 = \partial^2 \begin{pmatrix} -1 & ax \\ 0 & 0 \end{pmatrix} + \partial \begin{pmatrix} 0 & \frac{2}{a} \\ -\frac{2}{a} & 2x \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \frac{4}{a^2} \end{pmatrix},$$

$$E_3 = \partial^2 \begin{pmatrix} -ax & a^2x^2 - 1 \\ -1 & ax \end{pmatrix} + \partial \begin{pmatrix} -2a & 2x(a^2 + 2) \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \frac{2a^2+4}{a^2} \\ \frac{4}{a^2} & 0 \end{pmatrix},$$

$$E_4 = \partial^2 \begin{pmatrix} -ax & a^2x^2 + 1 \\ -1 & ax \end{pmatrix} + \partial \begin{pmatrix} -2a - \frac{4}{a} & 2a^2x + 4x \\ 0 & \frac{4}{a} \end{pmatrix} \\ + \begin{pmatrix} 0 & 2 + \frac{4}{a^2} \\ -\frac{4}{a^2} & 0 \end{pmatrix}.$$

*Muchas gracias por escuchar
:)*

- **Using quasi-Darboux transformation to construct matrix exceptional polynomials.** I. Bono Parisi, A. Durán, I. Zurrián. Preprint (2026).
- **A 3×3 singular solution to the Matrix Bochner Problem with non-polynomial algebra $\mathcal{D}(W)$.** I. Bono Parisi. *Journal of Approximation Theory* (2025).
- **Explicit construction of matrix-valued orthogonal polynomials of arbitrary size.** I. Bono Parisi. *Journal of Spectral Theory* (2025).
- **Structure of operator algebras for matrix orthogonal polynomials.** I. Bono Parisi, I. Pacharoni. *Numerical Algorithms* (2025).
- **The algebra $\mathcal{D}(W)$ via strong Darboux transformations.** I. Bono Parisi, I. Pacharoni. *Journal of Mathematical Analysis and Applications* (2025).

- **Darboux transformations and the algebra $\mathcal{D}(W)$.** I. Bono Parisi, I. Pacharoni. *Linear Algebra and its Applications* (2025).
- **Singular solutions of the matrix Bochner problem: the N -dimensional cases.** I. Bono Parisi, I. Pacharoni.
- **Singular examples of the matrix Bochner problem.** I. Bono Parisi, I. Pacharoni. *Journal of Approximation Theory* (2024).

Boundary value problems via weights

Fernando Ballesta-Yagüe

Universidad Complutense de Madrid

EARCO 2026, Universidad de La Rioja

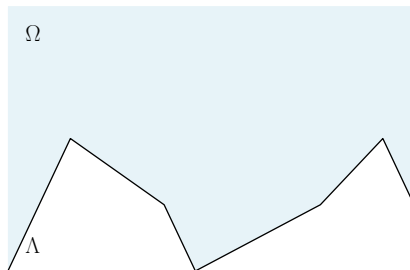
Joint work with María J. Carro

Problem

Goal: to use **weights** to study BVPs for Laplace's equation

$$\Delta v(z) = 0, \quad z \in \Omega$$

in graph Lipschitz domains in the plane with rough data:



Graph Lipschitz Domain

Contenido

- 1 The Dirichlet problem
- 2 The Neumann problem: L^p -solvability
- 3 The Neumann problem: $L^{p,1}$ -solvability

The Dirichlet problem

Let $g \in L^p(\partial\Omega, ds)$, $g: \partial\Omega \rightarrow \mathbb{R}$. We want to solve

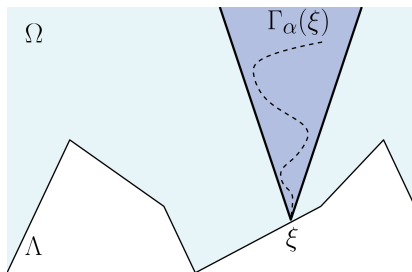
$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ v = g & \text{on } \partial\Omega. \end{cases}$$

The Dirichlet problem

Let $g \in L^p(\partial\Omega, ds)$, $g: \partial\Omega \rightarrow \mathbb{R}$. We want to solve

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ v = g & \text{on } \partial\Omega. \end{cases}$$

What does $v = g$ on $\partial\Omega$ means?



Non-tangential convergence

Additional estimate

L^p -solvability of the Dirichlet problem

Fix Ω and $p \in (1, \infty)$. Can we solve

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ v = g & \text{on } \partial\Omega, \\ \|\mathcal{M}_\alpha(v)\|_{L^p(\partial\Omega, ds)} \lesssim_\alpha \|g\|_{L^p(\partial\Omega, ds)} \end{cases}$$

for every $g \in L^p(\partial\Omega, ds)$?

Additional estimate

L^p -solvability of the Dirichlet problem

Fix Ω and $p \in (1, \infty)$. Can we solve

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ v = g & \text{on } \partial\Omega, \\ \|\mathcal{M}_\alpha(v)\|_{L^p(\partial\Omega, ds)} \lesssim_\alpha \|g\|_{L^p(\partial\Omega, ds)} \end{cases}$$

for every $g \in L^p(\partial\Omega, ds)$?

Here,

$$\mathcal{M}_\alpha(v)(\xi) := \sup_{z \in \Gamma_\alpha(\xi)} |v(z)|, \quad \xi \in \partial\Omega,$$

is the non-tangential maximal operator.

Additional estimate

L^p -solvability of the Dirichlet problem

Fix Ω and $p \in (1, \infty)$. Can we solve

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ v = g & \text{on } \partial\Omega, \\ \|\mathcal{M}_\alpha(v)\|_{L^p(\partial\Omega, ds)} \lesssim_\alpha \|g\|_{L^p(\partial\Omega, ds)} \end{cases}$$

for every $g \in L^p(\partial\Omega, ds)$?

Theorem (Kenig '80)

$\exists \varepsilon = \varepsilon(\Omega) > 0$ such that the Dirichlet problem is solvable in $L^p(\partial\Omega, ds)$ for every $p \in (2 - \varepsilon, \infty)$.

Idea: conformal mapping

Notice that

$$\Delta v = 0 \text{ in } \Omega \iff \Delta(v \circ \Phi) = 0 \text{ in } \mathbb{R}_+^2,$$

Idea: conformal mapping

Notice that

$$\Delta v = 0 \text{ in } \Omega \iff \Delta(v \circ \Phi) = 0 \text{ in } \mathbb{R}_+^2,$$

and

$$v = g \text{ on } \partial\Omega \iff v \circ \Phi = g \circ \Phi \text{ on } \partial\mathbb{R}_+^2.$$

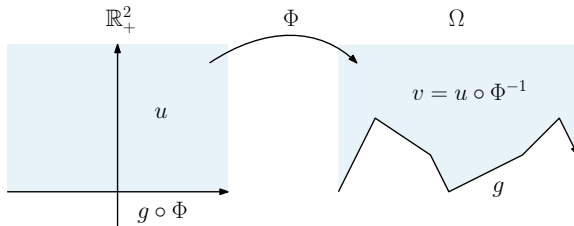
Idea: conformal mapping

Notice that

$$\Delta v = 0 \text{ in } \Omega \iff \Delta(v \circ \Phi) = 0 \text{ in } \mathbb{R}_+^2,$$

and

$$v = g \text{ on } \partial\Omega \iff v \circ \Phi = g \circ \Phi \text{ on } \partial\mathbb{R}_+^2.$$



Equivalent Dirichlet problems

Equivalent problems

So

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ v = g & \text{on } \partial\Omega, \\ \|\mathcal{M}_\alpha(v)\|_{L^p(\partial\Omega, ds)} \lesssim \|g\|_{L^p(\partial\Omega, ds)}, \end{cases}$$

if and only if $v = u \circ \Phi^{-1}$ with

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}_+^2, \\ u = g \circ \Phi & \text{on } \mathbb{R}, \\ \|\mathcal{M}_\alpha(u)\|_{L^p(\mathbb{R}, |\Phi'|)} \lesssim \|g \circ \Phi\|_{L^p(\mathbb{R}, |\Phi'|)}. \end{cases}$$

Equivalent problems

So

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ v = g & \text{on } \partial\Omega, \\ \|\mathcal{M}_\alpha(v)\|_{L^p(\partial\Omega, ds)} \lesssim \|g\|_{L^p(\partial\Omega, ds)}, \end{cases}$$

if and only if $v = u \circ \Phi^{-1}$ with

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}_+^2, \\ u = g \circ \Phi & \text{on } \mathbb{R}, \\ \|\mathcal{M}_\alpha(u)\|_{L^p(\mathbb{R}, |\Phi'|)} \lesssim \|g \circ \Phi\|_{L^p(\mathbb{R}, |\Phi'|)}. \end{cases}$$

The natural candidate is

$$u = P_y * (g \circ \Phi)(x), \quad (x, y) \in \mathbb{R}_+^2.$$

Problem in the upper-half plane

For this solution, we have

$$\mathcal{M}_\alpha(P_y * (g \circ \Phi)) \lesssim_\alpha \mathcal{M}_{hl}(g \circ \Phi)(x).$$

Problem in the upper-half plane

For this solution, we have

$$\mathcal{M}_\alpha(P_y * (g \circ \Phi)) \lesssim_\alpha \mathcal{M}_{hl}(g \circ \Phi)(x).$$

So we reduce the problem to the estimate:

$$\|\mathcal{M}_{hl}(g \circ \Phi)\|_{L^p(\mathbb{R}, |\Phi'|)} \lesssim \|g \circ \Phi\|_{L^p(\mathbb{R}, |\Phi'|)}.$$

Problem in the upper-half plane

For this solution, we have

$$\mathcal{M}_\alpha(P_y * (g \circ \Phi)) \lesssim_\alpha \mathcal{M}_{hl}(g \circ \Phi)(x).$$

So we reduce the problem to the estimate:

$$\|\mathcal{M}_{hl}(g \circ \Phi)\|_{L^p(\mathbb{R}, |\Phi'|)} \lesssim \|g \circ \Phi\|_{L^p(\mathbb{R}, |\Phi'|)}.$$

Here comes **weighted theory**.

Crash course on weighted theory

The Hardy-Littlewood maximal operator is defined as

$$\mathcal{M}_{hl}f(x) := \sup_{I \ni x} \frac{1}{|I|} \int_I |f(y)| dy.$$

Crash course on weighted theory

The Hardy-Littlewood maximal operator is defined as

$$\mathcal{M}_{hl}f(x) := \sup_{I \ni x} \frac{1}{|I|} \int_I |f(y)| dy.$$

Well-known:

$$\mathcal{M}_{hl}: L^p \rightarrow L^p, \quad p \in (1, \infty).$$

Crash course on weighted theory

The Hardy-Littlewood maximal operator is defined as

$$\mathcal{M}_{hl}f(x) := \sup_{I \ni x} \frac{1}{|I|} \int_I |f(y)| dy.$$

Well-known:

$$\mathcal{M}_{hl}: L^p(w) \rightarrow L^p(w), \quad p \in (1, \infty).$$

Crash course on weighted theory

The Hardy-Littlewood maximal operator is defined as

$$\mathcal{M}_{hl}f(x) := \sup_{I \ni x} \frac{1}{|I|} \int_I |f(y)| dy.$$

Well-known:

$$\mathcal{M}_{hl}: L^p(w) \rightarrow L^p(w), \quad p \in (1, \infty).$$

We define A_p as the class of weights w such that

$$\begin{aligned} w \in A_p : & \iff \mathcal{M}_{hl}: L^p(w) \rightarrow L^p(w) \\ & \iff \sup_Q \frac{1}{|Q|} \|\mathbb{1}_Q\|_{L^p(w)} \|\mathbb{1}_Q w^{-1}\|_{L^{p'}(w)} < \infty. \end{aligned}$$

Back to the problem

So if we want to have

$$\|\mathcal{M}_{hl}(g)\|_{L^p(\mathbb{R}, |\Phi'|)} \lesssim \|g\|_{L^p(\mathbb{R}, |\Phi'|)}$$

for every $g \in L^p(\mathbb{R}, |\Phi'|)$ we need $|\Phi'| \in A_p$.

Back to the problem

So if we want to have

$$\|\mathcal{M}_{hl}(g)\|_{L^p(\mathbb{R}, |\Phi'|)} \lesssim \|g\|_{L^p(\mathbb{R}, |\Phi'|)}$$

for every $g \in L^p(\mathbb{R}, |\Phi'|)$ we need $|\Phi'| \in A_p$.

To sum up:

If $|\Phi'| \in A_p \implies$ Dirichlet problem solvable in $L^p(\Lambda, ds)$.

Crash course on weights: continuation

- It is known that

$$|\Phi'| \in A_2.$$

Crash course on weights: continuation

- It is known that

$$|\Phi'| \in A_2.$$

- The A_p -classes are **nested**:

$$A_1 \subset A_{3/2} \subset A_2 \subset \dots$$

Crash course on weights: continuation

- It is known that

$$|\Phi'| \in A_2.$$

- The A_p -classes are **nested**:

$$A_1 \subset A_{3/2} \subset A_2 \subset \dots$$

Hence

$$|\Phi'| \in A_2 \subset A_p \quad \forall p \geq 2.$$

Crash course on weights: continuation

- It is known that

$$|\Phi'| \in A_2.$$

- The A_p -classes are **nested**:

$$A_1 \subset A_{3/2} \subset A_2 \subset \dots$$

Hence

$$|\Phi'| \in A_2 \subset A_p \quad \forall p \geq 2.$$

- The A_p -classes are **self-improving**: if $w \in A_p$, there exists $\varepsilon > 0$ such that $w \in A_{p-\varepsilon}$.

Crash course on weights: continuation

- It is known that

$$|\Phi'| \in A_2.$$

- The A_p -classes are **nested**:

$$A_1 \subset A_{3/2} \subset A_2 \subset \dots$$

Hence

$$|\Phi'| \in A_2 \subset A_p \quad \forall p \geq 2.$$

- The A_p -classes are **self-improving**: if $w \in A_p$, there exists $\varepsilon > 0$ such that $w \in A_{p-\varepsilon}$. Hence, there exists $\varepsilon > 0$ such that

$$|\Phi'| \in A_{2-\varepsilon}.$$

Crash course on weights: continuation

- It is known that

$$|\Phi'| \in A_2.$$

- The A_p -classes are **nested**:

$$A_1 \subset A_{3/2} \subset A_2 \subset \dots$$

Hence

$$|\Phi'| \in A_2 \subset A_p \quad \forall p \geq 2.$$

- The A_p -classes are **self-improving**: if $w \in A_p$, there exists $\varepsilon > 0$ such that $w \in A_{p-\varepsilon}$. Hence, there exists $\varepsilon > 0$ such that

$$|\Phi'| \in A_{2-\varepsilon} \subset A_2 \subset \dots$$

Theorem (Kenig, 1970s)

Let Ω be a graph Lipschitz domain. There exists $\varepsilon = \varepsilon(\Omega) > 0$ such that the Dirichlet problem is solvable in $L^p(\Lambda, ds)$ for every $p \in (2 - \varepsilon, \infty)$.

More precisely, it is solvable in L^p for every $p > p_\Omega$, where

$$p_\Omega := \inf\{q \in [1, \infty) : |\Phi'| \in A_q\}.$$

Endpoint results: Lorentz spaces

We have

$$L^{p,1} \subset L^p = L^{p,p} \subset L^{p,\infty}.$$

Endpoint results: Lorentz spaces

We have

$$L^{p,1} \subset L^p = L^{p,p} \subset L^{p,\infty}.$$

What are the weights w such that

$$\mathcal{M}_H: L^{p,1}(w) \rightarrow L^{p,\infty}(w)?$$

Endpoint results: Lorentz spaces

We have

$$L^{p,1} \subset L^p = L^{p,p} \subset L^{p,\infty}.$$

What are the weights w such that

$$\mathcal{M}_{hl}: L^{p,1}(w) \rightarrow L^{p,\infty}(w)?$$

We obtain a new (bigger) class in this case:

$$\begin{aligned} w \in A_p^{\mathcal{R}} &\iff \mathcal{M}_{hl}: L^{p,1}(w) \rightarrow L^{p,\infty}(w) \\ &\iff \sup_Q \frac{1}{|Q|} \|\mathbb{1}_Q\|_{L^{p,1}(w)} \|\mathbb{1}_Q w^{-1}\|_{L^{p',\infty}(w)} < \infty. \end{aligned}$$

Dirichlet endpoint

We cannot solve the Dirichlet problem in $L^p(\partial\Omega, ds)$ for $p = p_\Omega = 2 - \varepsilon$.

Dirichlet endpoint

We cannot solve the Dirichlet problem in $L^p(\partial\Omega, ds)$ for $p = p_\Omega = 2 - \varepsilon$. Can we solve it in the smaller space $L^{p,1}(\partial\Omega, ds)$?

Dirichlet endpoint

We cannot solve the Dirichlet problem in $L^p(\partial\Omega, ds)$ for $p = p_\Omega = 2 - \varepsilon$. Can we solve it in the smaller space $L^{p,1}(\partial\Omega, ds)$?

Theorem (Carro, Ortiz-Caraballo, '18)

Let Ω be a graph Lipschitz domain. Given $p \in (1, \infty)$, the Dirichlet problem is solvable in $L^{p,1}(\partial\Omega, ds)$ if $|\Phi'| \in A_p^{\mathcal{R}}$.

Dirichlet endpoint

We cannot solve the Dirichlet problem in $L^p(\partial\Omega, ds)$ for $p = p_\Omega = 2 - \varepsilon$. Can we solve it in the smaller space $L^{p,1}(\partial\Omega, ds)$?

Theorem (Carro, Ortiz-Caraballo, '18)

Let Ω be a graph Lipschitz domain. Given $p \in (1, \infty)$, the Dirichlet problem is solvable in $L^{p,1}(\partial\Omega, ds)$ if $|\Phi'| \in A_p^{\mathcal{R}}$.

This condition is satisfied only in some cases.

Recap

The general strategy is the following:

- 1 Find a solution in $\mathbb{R}_+^2$ to your boundary value problem.
- 2 Estimate the non-tangential maximal operator of this solution in the function space of interest.
- 3 Transfer the estimate to the graph Lipschitz domain.

Contenido

- 1 The Dirichlet problem
- 2 The Neumann problem: L^p -solvability
- 3 The Neumann problem: $L^{p,1}$ -solvability

The Neumann problem

$L^p(\partial\Omega, ds)$ -Neumann problem:

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ \partial_{\vec{n}} v = f & \text{on } \partial\Omega, \\ \|\mathcal{M}_\alpha(\nabla v)\|_{L^p(\partial\Omega, ds)} \lesssim \|f\|_{L^p(\partial\Omega, ds)}. \end{cases}$$

The Neumann problem

$L^p(\partial\Omega, ds)$ -Neumann problem:

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ \partial_{\vec{n}} v = f & \text{on } \partial\Omega, \\ \|\mathcal{M}_\alpha(\nabla v)\|_{L^p(\partial\Omega, ds)} \lesssim \|f\|_{L^p(\partial\Omega, ds)}. \end{cases}$$

Theorem (Dahlberg, Jerison, Kenig, '80s)

If Ω is a Lipschitz domain, there exists $\varepsilon > 0$ such that the Neumann problem is solvable in L^p for every $p \in (1, 2 + \varepsilon)$.

The Neumann problem

$L^p(\partial\Omega, ds)$ -Neumann problem:

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ \partial_{\vec{n}} v = f & \text{on } \partial\Omega, \\ \|\mathcal{M}_\alpha(\nabla v)\|_{L^p(\partial\Omega, ds)} \lesssim \|f\|_{L^p(\partial\Omega, ds)}. \end{cases}$$

Theorem (Dahlberg, Jerison, Kenig, '80s)

If Ω is a Lipschitz domain, there exists $\varepsilon > 0$ such that the Neumann problem is solvable in L^p for every $p \in (1, 2 + \varepsilon)$.

What is the weighted-theory reason behind this?

The Neumann problem

$L^p(\partial\Omega, d\nu)$ -Neumann problem:

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ \partial_{\vec{n}} v = f & \text{on } \partial\Omega, \\ \|\mathcal{M}_\alpha(\nabla v)\|_{L^p(\partial\Omega, d\nu)} \lesssim \|f\|_{L^p(\partial\Omega, d\nu)}. \end{cases}$$

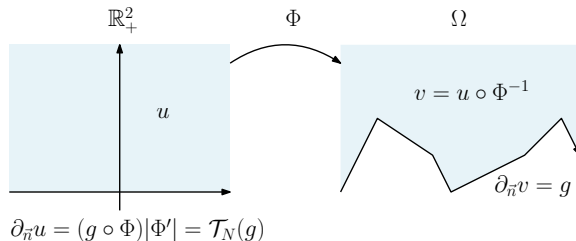
Theorem (Dahlberg, Jerison, Kenig, '80s)

If Ω is a Lipschitz domain, there exists $\varepsilon > 0$ such that the Neumann problem is solvable in L^p for every $p \in (1, 2 + \varepsilon)$.

What is the weighted-theory reason behind this?

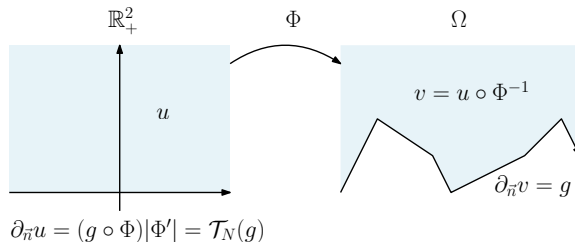
Corresponding problem

The corresponding problem in $\mathbb{R}_+^2$ is



Corresponding problem

The corresponding problem in $\mathbb{R}_+^2$ is

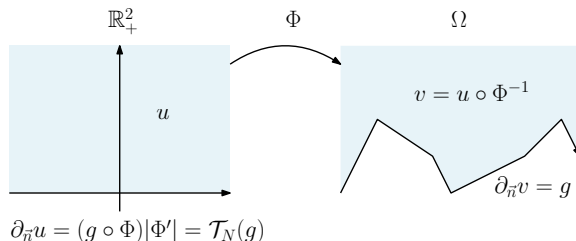


The norm of the datum on $\mathbb{R}_+^2$ is

$$\|f\|_{L^p(\partial\Omega, d\nu)} = \|f \circ \Phi\|_{L^p(\mathbb{R}, |\Phi'|(\nu \circ \Phi))} = \|(f \circ \Phi) |\Phi'|\|_{L^p(\mathbb{R}, |\Phi'|^{1-p}(\nu \circ \Phi))}.$$

Corresponding problem

The corresponding problem in $\mathbb{R}_+^2$ is



So it seems like we need to solve the following problem on $\mathbb{R}_+^2$:

$$\begin{cases} \Delta u = 0 \text{ in } \mathbb{R}_+^2, \\ \partial_{\bar{n}}u = g \text{ on } \mathbb{R}, \\ \|\mathcal{M}_\alpha(\nabla u)\|_{L^p(\mathbb{R}, |\Phi'|^{1-p}(\nu \circ \Phi))} \lesssim \|f\|_{L^p(\mathbb{R}, |\Phi'|^{1-p}(\nu \circ \Phi))}. \end{cases}$$

Candidate for a solution: Neumann integral

The solution now is given by the Neumann integral:

$$u_{g,N}(y_1, y_2) = -\frac{1}{\pi} \int_{\mathbb{R}} \log \left(\frac{\sqrt{(y_1 - t)^2 + y_2^2}}{1 + |t|} \right) g(t) dt, \quad (y_1, y_2) \in \mathbb{R}_+^2.$$

Candidate for a solution: Neumann integral

The solution now is given by the Neumann integral:

$$u_{g,N}(y_1, y_2) = -\frac{1}{\pi} \int_{\mathbb{R}} \log \left(\frac{\sqrt{(y_1 - t)^2 + y_2^2}}{1 + |t|} \right) g(t) dt, \quad (y_1, y_2) \in \mathbb{R}_+^2.$$

We have

$$\nabla u_{g,N}(x, y) = (Q_y * g(x), -P_y * g(x)),$$

Candidate for a solution: Neumann integral

The solution now is given by the Neumann integral:

$$u_{g,N}(y_1, y_2) = -\frac{1}{\pi} \int_{\mathbb{R}} \log \left(\frac{\sqrt{(y_1 - t)^2 + y_2^2}}{1 + |t|} \right) g(t) dt, \quad (y_1, y_2) \in \mathbb{R}_+^2.$$

We have

$$\nabla u_{g,N}(x, y) = (Q_y * g(x), -P_y * g(x)),$$

We need to control

$$\mathcal{M}_\alpha(|\nabla u_{g,N}|).$$

Candidate for a solution: Neumann integral

The solution now is given by the Neumann integral:

$$u_{g,N}(y_1, y_2) = -\frac{1}{\pi} \int_{\mathbb{R}} \log \left(\frac{\sqrt{(y_1 - t)^2 + y_2^2}}{1 + |t|} \right) g(t) dt, \quad (y_1, y_2) \in \mathbb{R}_+^2.$$

We have

$$\nabla u_{g,N}(x, y) = (Q_y * g(x), -P_y * g(x)),$$

We need to control

$$\mathcal{M}_\alpha(|\nabla u_{g,N}|).$$

Before we controled $\mathcal{M}_\alpha$ by $\mathcal{M}_{hl}$.

Candidate for a solution: Neumann integral

The solution now is given by the Neumann integral:

$$u_{g,N}(y_1, y_2) = -\frac{1}{\pi} \int_{\mathbb{R}} \log \left(\frac{\sqrt{(y_1 - t)^2 + y_2^2}}{1 + |t|} \right) g(t) dt, \quad (y_1, y_2) \in \mathbb{R}_+^2.$$

We have

$$\nabla u_{g,N}(x, y) = (Q_y * g(x), -P_y * g(x)),$$

We need to control

$$\mathcal{M}_\alpha(|\nabla u_{g,N}|).$$

Before we controled $\mathcal{M}_\alpha$ by $\mathcal{M}_{hl}$. Now we need something else...

Control of the gradient of $u_{g,N}$

Theorem (Carro, Naibo, Ortiz-Caraballo, '23)

Let $\eta \in (0, 1)$. Given $g \in L^1_c(\mathbb{R})$, there exists an η -sparse family of cubes $\mathcal{S}$ (depending on g) such that

$$\mathcal{M}_\alpha(|\nabla u_{N,g}|) \lesssim \mathcal{A}_{\mathcal{S}} g$$

Control of the gradient of $u_{g,N}$

Theorem (Carro, Naibo, Ortiz-Caraballo, '23)

Let $\eta \in (0, 1)$. Given $g \in L^1_c(\mathbb{R})$, there exists an η -sparse family of cubes $\mathcal{S}$ (depending on g) such that

$$\mathcal{M}_\alpha(|\nabla u_{N,g}|) \lesssim \mathcal{A}_S g$$

Here,

$$\mathcal{A}_S g(x) = \sum_{Q \in \mathcal{S}} \langle |g| \rangle_Q \mathbb{1}_Q.$$

End of the proof

If

$$|\Phi'|^{1-p}(\nu \circ \Phi) \in A_p,$$

then

$$\begin{aligned}\|\mathcal{M}_\alpha(\nabla u_{N,g})\|_{L^p(\mathbb{R}, |\Phi'|^{1-p}(\nu \circ \Phi))} &\lesssim \|\mathcal{A}_S g\|_{L^p(\mathbb{R}, |\Phi'|^{1-p}(\nu \circ \Phi))} \\ &\lesssim \|g\|_{L^p(\mathbb{R}, |\Phi'|^{1-p}(\nu \circ \Phi))}.\end{aligned}$$

End of the proof

If

$$|\Phi'|^{1-p}(\nu \circ \Phi) \in A_p,$$

then

$$\begin{aligned} \|\mathcal{M}_\alpha(\nabla u_{N,g})\|_{L^p(\mathbb{R}, |\Phi'|^{1-p}(\nu \circ \Phi))} &\lesssim \|\mathcal{A}_S g\|_{L^p(\mathbb{R}, |\Phi'|^{1-p}(\nu \circ \Phi))} \\ &\lesssim \|g\|_{L^p(\mathbb{R}, |\Phi'|^{1-p}(\nu \circ \Phi))}. \end{aligned}$$

The set of p 's for which the Neumann problem is solvable in $L^p(\partial\Omega, \nu)$ is

$$R(\nu) := \{p \in (1, \infty) : |\Phi'|^{1-p}(\nu \circ \Phi) \in A_p\}.$$

Case of arc-length measure: $\nu = 1$

If $\nu \equiv 1$, then

$$|\Phi'|^{1-p}(\nu \circ \Phi) = |\Phi'|^{1-p} \in A_p \iff |\Phi'| \in A_{p'},$$

Case of arc-length measure: $\nu = 1$

If $\nu \equiv 1$, then

$$|\Phi'|^{1-p}(\nu \circ \Phi) = |\Phi'|^{1-p} \in A_p \iff |\Phi'| \in A_{p'},$$

This holds for

$$p' \in (2 - \varepsilon', \infty) \iff p \in (1, 2 + \varepsilon),$$

as we wanted to see.

Case of arc-length measure: $\nu = 1$

If $\nu \equiv 1$, then

$$|\Phi'|^{1-p}(\nu \circ \Phi) = |\Phi'|^{1-p} \in A_p \iff |\Phi'| \in A_{p'},$$

This holds for

$$p' \in (2 - \varepsilon', \infty) \iff p \in (1, 2 + \varepsilon),$$

as we wanted to see.

So

$$R(1) = \{p \in (1, \infty) : |\Phi'|^{1-p} \in A_p\} = (1, (p_\Omega)').$$

Contenido

- 1 The Dirichlet problem
- 2 The Neumann problem: L^p -solvability
- 3 The Neumann problem: $L^{p,1}$ -solvability

Range of solvability

The set of p 's for which the Neumann problem is solvable in $L^p(\partial\Omega, \nu)$ is

$$R(\nu) := \{p \in (1, \infty) : |\Phi'|^{1-p}(\nu \circ \Phi) \in A_p\}.$$

Range of solvability

The set of p 's for which the Neumann problem is solvable in $L^p(\partial\Omega, \nu)$ is

$$R(\nu) := \{p \in (1, \infty) : |\Phi'|^{1-p}(\nu \circ \Phi) \in A_p\}.$$

We remark that:

Range of solvability

The set of p 's for which the Neumann problem is solvable in $L^p(\partial\Omega, \nu)$ is

$$R(\nu) := \{p \in (1, \infty) : |\Phi'|^{1-p}(\nu \circ \Phi) \in A_p\}.$$

We remark that:

- It may be empty.

Range of solvability

The set of p 's for which the Neumann problem is solvable in $L^p(\partial\Omega, \nu)$ is

$$R(\nu) := \{p \in (1, \infty) : |\Phi'|^{1-p}(\nu \circ \Phi) \in A_p\}.$$

We remark that:

- It may be empty.
- If $R(\nu) \neq \emptyset$, it is an interval (p_-, p_+) .

Range of solvability

The set of p 's for which the Neumann problem is solvable in $L^p(\partial\Omega, \nu)$ is

$$R(\nu) := \{p \in (1, \infty) : |\Phi'|^{1-p}(\nu \circ \Phi) \in A_p\}.$$

We remark that:

- It may be empty.
- If $R(\nu) \neq \emptyset$, it is an interval (p_-, p_+) .

Can we solve it at $p \in \{p_-, p_+\}$ in a smaller space than L^p ?

Range of solvability

The set of p 's for which the Neumann problem is solvable in $L^p(\partial\Omega, \nu)$ is

$$R(\nu) := \{p \in (1, \infty) : |\Phi'|^{1-p}(\nu \circ \Phi) \in A_p\}.$$

We remark that:

- It may be empty.
- If $R(\nu) \neq \emptyset$, it is an interval (p_-, p_+) .

Can we solve it at $p \in \{p_-, p_+\}$ in a smaller space than L^p ?

We distinguish two cases:

$$p_- = 1 \quad \text{and} \quad p_- > 1.$$

$L^{p,1}(\Lambda, \nu)$ -solvability

Consider

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ \partial_{\vec{n}} v = g & \text{on } \partial\Omega, \\ \|\mathcal{M}_\alpha(\nabla v)\|_{L^{p,\infty}(\partial\Omega,\nu)} \lesssim \|g\|_{L^{p,1}(\partial\Omega,\nu)}. \end{cases}$$

The norm of the boundary datum can be written as

$$\|g\|_{L^{p,1}(\partial\Omega,\nu)} = \|g \circ \Phi\|_{L^{p,1}(\mathbb{R}, |\Phi'| \nu \circ \Phi)} = \left\| \frac{\mathcal{T}_N(g)}{|\Phi'|} \right\|_{L^{p,1}(\mathbb{R}, \Phi(\nu))}.$$

Estimate on $\mathbb{R}$

It turns out that the estimate that we need to have in $\mathbb{R}_+^2$ is

$$\left\| \frac{\mathcal{A}_S(\mathcal{T}_N(g))}{|\Phi'|} \right\|_{L^{p,\infty}(\mathbb{R}, \Phi(\nu))} \lesssim \left\| \frac{\mathcal{T}_N(g)}{|\Phi'|} \right\|_{L^{p,1}(\mathbb{R}, \Phi(\nu))}$$

Estimate on $\mathbb{R}$

It turns out that the estimate that we need to have in $\mathbb{R}_+^2$ is

$$\left\| \frac{\mathcal{A}_S(\mathcal{T}_N(g))}{|\Phi'|} \right\|_{L^{p,\infty}(\mathbb{R}, \Phi(\nu))} \lesssim \left\| \frac{\mathcal{T}_N(g)}{|\Phi'|} \right\|_{L^{p,1}(\mathbb{R}, \Phi(\nu))}$$

Definition (B.Y.-Carro '25)

We say that $(u, \nu) \in \mathcal{S}_p^{\mathcal{R}}$ if

$$\left\| \frac{\mathcal{A}_S(fu)}{u} \right\|_{L^{p,\infty}(\mathbb{R}, \nu)} \lesssim_{\eta, u, \nu} \|f\|_{L^{p,1}(\mathbb{R}, \nu)}$$

Estimate on $\mathbb{R}$

It turns out that the estimate that we need to have in $\mathbb{R}_+^2$ is

$$\left\| \frac{\mathcal{A}_S(\mathcal{T}_N(g))}{|\Phi'|} \right\|_{L^{p,\infty}(\mathbb{R}, \Phi(\nu))} \lesssim \left\| \frac{\mathcal{T}_N(g)}{|\Phi'|} \right\|_{L^{p,1}(\mathbb{R}, \Phi(\nu))}$$

Definition (B.Y.-Carro '25)

We say that $(u, \nu) \in \mathcal{S}_p^{\mathcal{R}}$ if

$$\left\| \frac{\mathcal{A}_S(fu)}{u} \right\|_{L^{p,\infty}(\mathbb{R}, \nu)} \lesssim_{\eta, u, \nu} \|f\|_{L^{p,1}(\mathbb{R}, \nu)}$$

We need

$$(|\Phi'|, \Phi(\nu)) \in \mathcal{S}_p^{\mathcal{R}}.$$

Solvability at $L^{p_{\pm},1}(\Lambda, d\nu)$

Theorem (B.Y.-Carro '25)

Let $\nu \in A_{\infty}(\Lambda)$ be such that

$$R(\nu) = (p_-, p_+) \neq \emptyset.$$

Let

$$p \in \{p_-, p_+\} \subset (1, \infty).$$

If

$$(|\Phi'|, \Phi(\nu)) \in \mathcal{S}_p^{\mathcal{R}},$$

then the Neumann problem is solvable in $X = L^{p,1}(\partial\Omega, d\nu)$ with $Y = L^{p,\infty}(\partial\Omega, d\nu)$.

Sufficient condition for $\mathcal{S}_p^{\mathcal{R}}$

Theorem (B.Y.-Carro '25)

Assume u and v are weights, with $v \in A_\infty$, such that

$$\sup_Q \frac{u(Q) \left(\frac{v}{u}\right)(Q)}{|Q|v(Q)} < \infty,$$

and such that

$$\frac{v}{u} \in A_p(u) \text{ and } u \in A_{p'}^{\mathcal{R}}\left(\frac{v}{u}\right), \quad \text{or} \quad \frac{v}{u} \in A_p^{\mathcal{R}}(u) \text{ and } u \in A_{p'}\left(\frac{v}{u}\right).$$

Then $(u, v) \in \mathcal{S}_p^{\mathcal{R}}$.

Sufficient condition for $\mathcal{S}_p^{\mathcal{R}}$

Theorem (B.Y.-Carro '25)

Assume u and v are weights, with $v \in A_\infty$, such that

$$\sup_Q \frac{u(Q) \left(\frac{v}{u}\right)(Q)}{|Q|v(Q)} < \infty,$$

and such that

$$\frac{v}{u} \in A_p(u) \text{ and } u \in A_{p'}^{\mathcal{R}}\left(\frac{v}{u}\right), \quad \text{or} \quad \frac{v}{u} \in A_p^{\mathcal{R}}(u) \text{ and } u \in A_{p'}\left(\frac{v}{u}\right).$$

Then $(u, v) \in \mathcal{S}_p^{\mathcal{R}}$.

Remark:

$$\frac{v}{u} \in A_p(u) \iff \mathcal{M}_u: L^p(v) \rightarrow L^p(v).$$

Sufficient condition for $\mathcal{S}_p^{\mathcal{R}}$

Theorem (B.Y.-Carro '25)

Assume u and v are weights, with $v \in A_\infty$, such that

$$\sup_Q \frac{u(Q) \left(\frac{v}{u}\right)(Q)}{|Q|v(Q)} < \infty,$$

and such that

$$\frac{v}{u} \in A_p(u) \text{ and } u \in A_{p'}^{\mathcal{R}}\left(\frac{v}{u}\right), \quad \text{or} \quad \frac{v}{u} \in A_p^{\mathcal{R}}(u) \text{ and } u \in A_{p'}\left(\frac{v}{u}\right).$$

Then $(u, v) \in \mathcal{S}_p^{\mathcal{R}}$.

If $v = u = |\Phi'|$, the conditions are $u \in A_{p'}^{\mathcal{R}}(\mathbb{R})$ or $u \in A_{p'}(\mathbb{R})$.

Sufficient condition for $\mathcal{S}_p^{\mathcal{R}}$

Theorem (B.Y.-Carro '25)

Assume u and v are weights, with $v \in A_\infty$, such that

$$\sup_Q \frac{u(Q) \left(\frac{v}{u}\right)(Q)}{|Q|v(Q)} < \infty,$$

and such that

$$\frac{v}{u} \in A_p(u) \text{ and } u \in A_{p'}^{\mathcal{R}}\left(\frac{v}{u}\right), \quad \text{or} \quad \frac{v}{u} \in A_p^{\mathcal{R}}(u) \text{ and } u \in A_{p'}\left(\frac{v}{u}\right).$$

Then $(u, v) \in \mathcal{S}_p^{\mathcal{R}}$.

If $v = u = |\Phi'|$, the conditions are $u \in A_{p'}^{\mathcal{R}}(\mathbb{R})$ or $u \in A_{p'}(\mathbb{R})$. So if $|\Phi'| \in A_{p\Omega}^{\mathcal{R}}$, the problem is solvable in $L^{p'_\Omega, 1}(\partial\Omega, ds)$.

Example: cone with power weights, endpoint case

Corollary (B.Y.-Carro '25)

The Neumann problem is solvable in $L^{p\pm,1}(\partial\Omega, \nu)$ for $\partial\Omega$ a cone of angle $\alpha\pi$ (with $\alpha \in (0, 2)$) and $\nu(\xi) = |\xi|^\beta$ if

$$\begin{cases} \alpha\beta > -1, \\ \beta > -1. \end{cases}$$

Some open questions

- Optimality of the conditions.

Some open questions

- Optimality of the conditions. For example, does $L^p(\Lambda, \nu)$ -solvability of the Neumann problem imply $|\Phi'|^{1-p}(\nu \circ \Phi) \in A_{p'}(\mathbb{R})$?

Some open questions

- Optimality of the conditions. For example, does $L^p(\Lambda, \nu)$ -solvability of the Neumann problem imply $|\Phi'|^{1-p}(\nu \circ \Phi) \in A_{p'}(\mathbb{R})$?
- Cancellative sparse domination for $\mathcal{M}_\alpha(\nabla u_{g,N})$, where,

$$|\nabla u_{g,N}(x, y)| = \sqrt{(Q_y * g(x))^2 + (P_y * g(x))^2}.$$

Some open questions

- Optimality of the conditions. For example, does $L^p(\Lambda, \nu)$ -solvability of the Neumann problem imply $|\Phi'|^{1-p}(\nu \circ \Phi) \in A_{p'}(\mathbb{R})$?
- Cancellative sparse domination for $\mathcal{M}_\alpha(\nabla u_{g,N})$, where,

$$|\nabla u_{g,N}(x, y)| = \sqrt{(Q_y * g(x))^2 + (P_y * g(x))^2}.$$

This would give another proof for H_{at}^1 -solvability.

Some open questions

- Optimality of the conditions. For example, does $L^p(\Lambda, \nu)$ -solvability of the Neumann problem imply $|\Phi'|^{1-p}(\nu \circ \Phi) \in A_{p'}(\mathbb{R})$?
- Cancellative sparse domination for $\mathcal{M}_\alpha(\nabla u_{g,N})$, where,

$$|\nabla u_{g,N}(x, y)| = \sqrt{(Q_y * g(x))^2 + (P_y * g(x))^2}.$$

This would give another proof for H_{at}^1 -solvability.

- Characterisation of $\mathcal{S}_p^{\mathcal{R}}$.

¡Gracias!

¡Gracias!

HEAT SMOOTHING FOR THE WASSERSTEIN DISTANCE AND RANDOM NORMAL MATRICES

ENCUENTROS DE ANÁLISIS REAL Y COMPLEJO
(EARCO 2026)

PABLO GARCÍA ARIAS

UNIVERSITAT DE BARCELONA

MAYO 2026

Proyecto PID2021-123405NB-I00 financiado por:



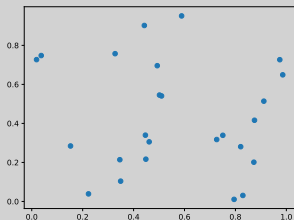
DETERMINANTAL POINT PROCESSES

- A point process is a random variable that outputs a discrete set of points.

DETERMINANTAL POINT PROCESSES

- A point process is a random variable that outputs a discrete set of points.
- They can be used to produce Monte-Carlo methods:

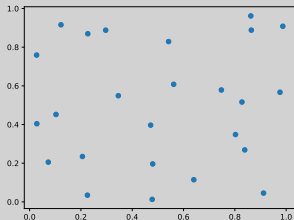
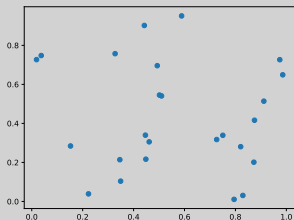
$$\int f(x) dx \approx \frac{1}{N} \sum f(x_n) \quad \mu_N = \frac{1}{N} \sum \delta_{x_n} \approx dx$$



DETERMINANTAL POINT PROCESSES

- A point process is a random variable that outputs a discrete set of points.
- We call it a Determinantal Point Process with kernel K when joint intensities are given by $\rho_K(x_1, \dots, x_k) = \det K(x_i, x_j)$.
- They can be used to produce Monte-Carlo methods:

$$\int f(x) dx \approx \frac{1}{N} \sum f(x_n) \qquad \mu_N = \frac{1}{N} \sum \delta_{x_n} \approx dx$$



RANDOM NORMAL MATRICES

- Fix a potential weight $Q : \mathbb{C} \rightarrow \mathbb{R}$.
- **RNM Model:** $N \times N$ complex normal matrices with the probability measure:

$$\frac{1}{Z_N} e^{-N \operatorname{Tr} Q(M)} dM$$

RANDOM NORMAL MATRICES

- Fix a potential weight $Q : \mathbb{C} \rightarrow \mathbb{R}$.
- **RNM Model:** $N \times N$ complex normal matrices with the probability measure:

$$\frac{1}{Z_N} e^{-N \operatorname{Tr} Q(M)} dM$$

- The eigenvalues form a projection DPP with kernel

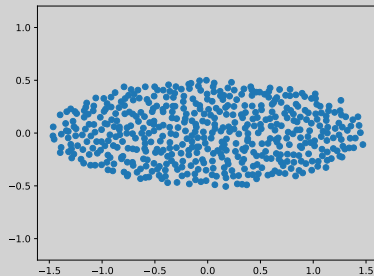
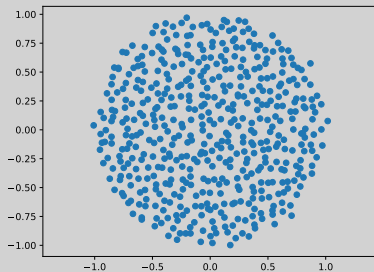
$$K_N(z, w) = e^{-\frac{N}{2}(Q(z)+Q(w))} \sum_{j=0}^{N-1} p_j(z) \overline{p_j(w)}$$

for the orthonormal polynomials

$$\int p_j(z) \overline{p_k(z)} e^{-NQ(z)} dA(z) = \delta_{j,k}.$$

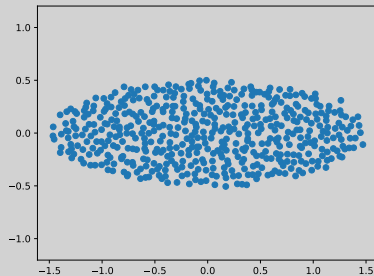
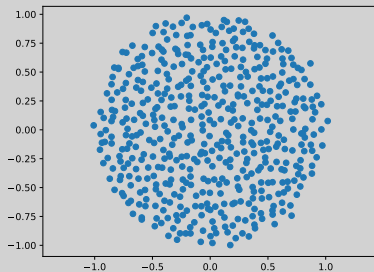
DROPLET AND EQUILIBRIUM MEASURE

- The spectrum tends to accumulate in a compact set called the droplet S .



DROPLET AND EQUILIBRIUM MEASURE

- The spectrum tends to accumulate in a compact set called the droplet S .



- The empirical measure converge weakly to $\sigma = \frac{1}{4}\Delta Q\chi_S dA$.
- The Ginibre ensemble is the simplest of these models:
 $Q(z) = |z|^2, \sigma = \frac{1}{\pi}\chi_{\mathbb{D}}.$

WASSERSTEIN DISTANCE

Wasserstein distance

Given two probability measures μ, ν the Wasserstein distance is

$$W_2(\mu, \nu) = \left(\inf \left\{ \int d(x, y)^2 d\pi(x, y) : \pi \text{ has } \mu, \nu \text{ as marginals} \right\} \right)^{\frac{1}{2}}$$

WASSERSTEIN DISTANCE

Wasserstein distance

Given two probability measures μ, ν the Wasserstein distance is

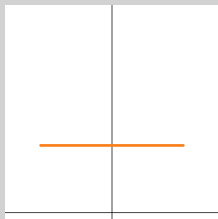
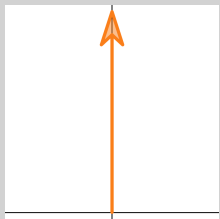
$$W_2(\mu, \nu) = \left(\inf \left\{ \int d(x, y)^2 d\pi(x, y) : \pi \text{ has } \mu, \nu \text{ as marginals} \right\} \right)^{\frac{1}{2}}$$

- This quantifies of the approximation in the sense:

$$W_2(\mu_N, d\sigma) \rightarrow 0 \iff \int f d\mu_N \rightarrow \int f d\sigma \quad \forall f \in C$$

- It has the lower bound $W_2(\mu_N, dx) \gtrsim N^{-\frac{1}{d}}$.
- For iid points in dimension 2, $\mathbb{E}W_2(\mu_N, dx) \sim \frac{\sqrt{\log N}}{\sqrt{N}}$

HEAT SMOOTHING



HEAT SMOOTHING

- One can perform a smoothing via the heat equation:



- Fix a convex set Ω . Let μ_t, ν_t be

$$\left. \begin{aligned} \partial_t \mu_t &= \Delta_x \mu_t & x \in \Omega \\ \partial_{\vec{n}} \mu_t &= 0 & x \in \partial\Omega \\ \mu_0 &= \mu \end{aligned} \right\}$$

HEAT SMOOTHING

- One can perform a smoothing via the heat equation:



- Fix a convex set Ω . Let μ_t, ν_t be

$$\left. \begin{aligned} \partial_t \mu_t &= \Delta_x \mu_t & x \in \Omega \\ \partial_{\vec{n}} \mu_t &= 0 & x \in \partial\Omega \\ \mu_0 &= \mu \end{aligned} \right\}$$

- Then we are going to consider

$$W_2(\mu, \nu) \leq W_2(\mu, \mu_t) + W_2(\mu_t, \nu_t) + W_2(\nu, \nu_t).$$

- If $P_t(x, y)$ is the heat kernel, then $\mu_t(x) = \int P_t(x, y) d\mu(y)$.
- It satisfies the following dispersion property:

$$\int_{\Omega} |x - y|^2 P_t(x, y) dy \lesssim t$$

- If $P_t(x, y)$ is the heat kernel, then $\mu_t(x) = \int P_t(x, y) d\mu(y)$.
- It satisfies the following dispersion property:

$$\int_{\Omega} |x - y|^2 P_t(x, y) dy \lesssim t$$

- This implies

$$W_2(\mu, \mu_t) + W_2(\nu, \nu_t) \lesssim t^{\frac{1}{2}}$$

- The main tool is Peyre's result relating W_2 with negative Sobolev norms

$$W_2(\mu_t, \nu_t) \leq 2 \sup \left\{ \left| \int f d(\mu_t - \nu_t) \right| : \int |\nabla f|^2 d\nu_t \leq 1 \right\}.$$

- The main tool is Peyre's result relating W_2 with negative Sobolev norms

$$W_2(\mu_t, \nu_t) \leq 2 \sup \left\{ \left| \int f d(\mu_t - \nu_t) \right| : \int |\nabla f|^2 d\nu_t \leq 1 \right\}.$$

- If we assume that $\nu(x) \gtrsim dx$ in Ω we can write

$$W_2(\mu_t, \nu_t) \lesssim \left(\sum_{k \geq 1} \frac{e^{-\lambda_k t}}{\lambda_k} |\mu(\phi_k) - \nu(\phi_k)|^2 \right)^{\frac{1}{2}}$$

- This is because the heat kernel can be written as $P_t(x, y) = \sum_{k \geq 0} e^{-t\lambda_k} \phi_k(x) \phi_k(y)$ for the Neumann eigenfunctions $\Delta \phi_k = -\lambda_k \phi_k$.

OBTAINED INEQUALITY

■ All in all we have seen

$$W_2(\mu, \nu) \lesssim t^{\frac{1}{2}} + \left(\sum_{k \geq 1} \frac{e^{-\lambda_k t}}{\lambda_k} |\mu(\phi_k) - \nu(\phi_k)|^2 \right)^{\frac{1}{2}}.$$

OBTAINED INEQUALITY

- All in all we have seen

$$W_2(\mu, \nu) \lesssim t^{\frac{1}{2}} + \left(\sum_{k \geq 1} \frac{e^{-\lambda_k t}}{\lambda_k} |\mu(\phi_k) - \nu(\phi_k)|^2 \right)^{\frac{1}{2}}.$$

- When there are random points, we can pass to the expectation:

$$\mathbb{E}W_2(\mu_N, \nu) \lesssim t^{\frac{1}{2}} + \left(\sum_{k \geq 1} \frac{e^{-\lambda_k t}}{\lambda_k} \mathbb{E}|\mu_N(\phi_k) - \nu(\phi_k)|^2 \right)^{\frac{1}{2}}.$$

OBTAINED INEQUALITY

- All in all we have seen

$$W_2(\mu, \nu) \lesssim t^{\frac{1}{2}} + \left(\sum_{k \geq 1} \frac{e^{-\lambda_k t}}{\lambda_k} |\mu(\phi_k) - \nu(\phi_k)|^2 \right)^{\frac{1}{2}}.$$

- When there are random points, we can pass to the expectation:

$$\mathbb{E}W_2(\mu_N, \nu) \lesssim t^{\frac{1}{2}} + \left(\sum_{k \geq 1} \frac{e^{-\lambda_k t}}{\lambda_k} \mathbb{E}|\mu_N(\phi_k) - \nu(\phi_k)|^2 \right)^{\frac{1}{2}}.$$

- So we need to study

$$\mathbb{E}|\mu_N(\phi_k) - \nu(\phi_k)|^2 = \mathbb{E} \left| \int \phi_k d(\mu_N - \nu) \right|^2$$

HEAT SMOOTHING INEQUALITY

Theorem

Given a sequence of measures μ_N on a convex domain Ω , and a limit measure ν with same mass and $\nu \gtrsim dx$ on Ω , assume we have the bound

$$\mathbb{E}|\mu_N(\phi_k) - \nu(\phi_k)|^2 \lesssim \frac{\lambda_k^b}{N^a}.$$

Let $\gamma = \frac{a}{2b+d}$. Then

- $\mathbb{E}W_2(\mu_N, \nu) \lesssim \frac{1}{N^\gamma}$ if $b + \frac{d}{2} > 1$.
- $\mathbb{E}W_2(\mu_N, \nu) \lesssim \frac{\sqrt{\log N}}{N^\gamma}$ if $b + \frac{d}{2} = 1$.

HEAT SMOOTHING INEQUALITY

Theorem

Given a sequence of measures μ_N on a convex domain Ω , and a limit measure ν with same mass and $\nu \gtrsim dx$ on Ω , assume we have the bound

$$\mathbb{E}|\mu_N(\phi_k) - \nu(\phi_k)|^2 \lesssim \frac{\lambda_k^b}{N^a}.$$

Let $\gamma = \frac{a}{2b+d}$. Then

- $\mathbb{E}W_2(\mu_N, \nu) \lesssim \frac{1}{N^\gamma}$ if $b + \frac{d}{2} > 1$.
- $\mathbb{E}W_2(\mu_N, \nu) \lesssim \frac{\sqrt{\log N}}{N^\gamma}$ if $b + \frac{d}{2} = 1$.

We are going to use the bound with $b = 0$ and $a = 1$:

$$\mathbb{E}|\mu_N(\phi_k) - \nu(\phi_k)|^2 \lesssim \frac{1}{N}$$

Theorem

Let $z_1, \dots, z_N$ be given by the RNM model, for a suitable potential Q having a convex droplet S , then

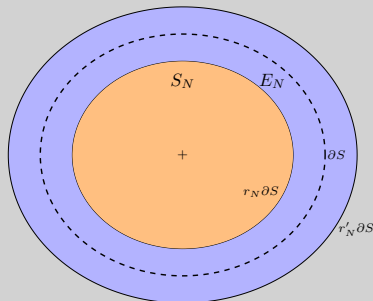
$$\mathbb{E}W_2(\mu_N, \sigma) \lesssim \frac{\sqrt{\log N}}{\sqrt{N}}.$$

Theorem

Let $z_1, \dots, z_N$ be given by the RNM model, for a suitable potential Q having a convex droplet S , then

$$\mathbb{E}W_2(\mu_N, \sigma) \lesssim \frac{\sqrt{\log N}}{\sqrt{N}}.$$

- Previous results were known for the Ginibre ensemble.
- Divide the plane in three regions:
 - ▶ The bulk S_N
 - ▶ The edge E_N
 - ▶ The outside region



- The points tend to accumulate in the droplet, so we expect very few points to be far away from it.
- In fact there is an exponential decay to the probability.
- It is easy to get

$$\mathbb{E}W_2 \left(\mu_N|_{\mathbb{C} \setminus (E_N \cup S_N)}, \mu_N(\mathbb{C} \setminus (E_N \cup S_N)) \frac{\sigma|_{E_N}}{\sigma(E_N)} \right) \lesssim \frac{1}{\sqrt{N}}$$

- This is the region where we have less tools to control the points.
- I have generalized a proof by Prod'Homme for the Ginibre ensemble, so that now works for general potentials.
- It involves iterative splittings of the edge, and comparing each piece with its two halves.
- We get

$$\mathbb{E}W_2 \left(\mu_N(E_N), \mu_N(E_N) \frac{\sigma|_{E_N}}{\sigma(E_N)} \right) \lesssim \frac{\sqrt{\log N}}{\sqrt{N}}$$

- In the bulk we have the uniform bound

$$|K(z, z) - N\sigma(z)| \leq C.$$

- In the bulk we have the uniform bound

$$|K(z, z) - N\sigma(z)| \leq C.$$

- We can apply the heat smoothing distinguishing two cases:
- CASE 1: There are few points $\#\{z_n \in S_N\} \leq \frac{1}{2}\mathbb{E}\#\{z_n \in S_N\}$.
- CASE 2: There are enough points to apply the heat smoothing between $\mu|_{S_N}$ and $\mu(S_N)^{\frac{\sigma|_{S_N}}{\sigma(S_N)}}$.

- In the bulk we have the uniform bound

$$|K(z, z) - N\sigma(z)| \leq C.$$

- We can apply the heat smoothing distinguishing two cases:
- CASE 1: There are few points $\#\{z_n \in S_N\} \leq \frac{1}{2}\mathbb{E}\#\{z_n \in S_N\}$.
- CASE 2: There are enough points to apply the heat smoothing between $\mu|_{S_N}$ and $\mu(S_N)^{\frac{\sigma|_{S_N}}{\sigma(S_N)}}$.

- ▶ Here we need to study the statistics

$$\mathbb{E} \left| \frac{1}{N} \sum_{x_n \in S_N} \phi_k(x_n) - \frac{\mu(S_N)}{\sigma(S_N)} \int \phi_k d\sigma \right|^2$$

$$\Rightarrow^* \text{Var} \left(\frac{1}{N} \sum_{x_n \in S_N} \phi_k(x_n) \right) + \left| \int_{S_N} \phi_k(z) \left(\frac{1}{N} K(z, z) - \sigma(z) \right) \right|^2 \lesssim \frac{1}{N}$$

- After bounding each part independently it is left to prove

$$\mathbb{E}W_2 \left(\frac{\mu(\mathbb{C} \setminus S_N)}{\sigma(E)} \sigma|_E + \frac{\mu(S_N)}{\sigma(S_N)} \sigma|_{S_N}, \sigma \right) \lesssim \frac{1}{\sqrt{N}}.$$

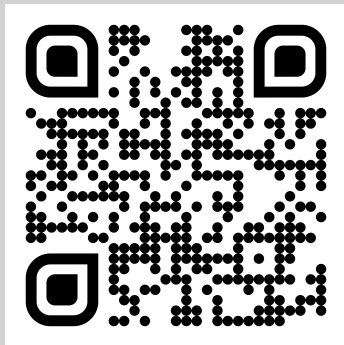
- After bounding each part independently it is left to prove

$$\mathbb{E}W_2 \left(\frac{\mu(\mathbb{C} \setminus S_N)}{\sigma(E)} \sigma|_E + \frac{\mu(S_N)}{\sigma(S_N)} \sigma|_{S_N}, \sigma \right) \lesssim \frac{1}{\sqrt{N}}.$$

- We “lose” the logarithmic factor in two places:
 - ▶ In the edge, due to not having rotational invariance.
 - ▶ In the bulk, due to the Neumann eigenfunctions having a sharp cut-off.

REFERENCE

- [1] *Quantitative equidistribution of eigenvalues of Random Normal Matrices in the Wasserstein distance*, P. García Arias, <https://arxiv.org/abs/2603.18313>



Breaking of Breathers by a Point Defect in the Sine-Gordon Equation.

EARCO 2026, Logroño

Sergio Moroni

Basque Center of Applied Mathematics

31 May 2026

(1)	(2)	(3)	(4)
 GOBIERNO VASCO GOVERNMENT OF THE BASQUE COUNTRY	iKUR estrategia	 Financiado por la Unión Europea NextGenerationEU	 Plan de Recuperación, Transformación y Resiliencia

Sine Gordon Equation

The sine–Gordon equation is a classical scalar field equation:

$$u_{tt} - u_{xx} + \sin(u) = 0 \quad t, x \in \mathbb{R} \quad (\text{SG})$$

The equation is known to be have integrable regularity (Lax–Pair formulation, Backlund transform, infinitely many conserved quantities, soliton resolution).

Sine Gordon Equation

The sine–Gordon equation is a classical scalar field equation:

$$u_{tt} - u_{xx} + \sin(u) = 0 \quad t, x \in \mathbb{R} \quad (\text{SG})$$

The equation is known to have integrable regularity (Lax–Pair formulation, Backlund transform, infinitely many conserved quantities, soliton resolution).

By Inverse Scattering one can construct explicit solutions as the breathers, localized in space and periodic in time:

$$B_{\beta}(x, t) = 4 \arctan \frac{\beta \cos(\alpha t)}{\alpha \cosh(\beta x)}; \quad \alpha^2 + \beta^2 = 1$$

Sine Gordon Equation

The sine–Gordon equation is a classical scalar field equation:

$$u_{tt} - u_{xx} + \sin(u) = 0 \quad t, x \in \mathbb{R} \quad (\text{SG})$$

The equation is known to have integrable regularity (Lax–Pair formulation, Backlund transform, infinitely many conserved quantities, soliton resolution).

By Inverse Scattering one can construct explicit solutions as the breathers, localized in space and periodic in time:

$$B_\beta(x, t) = 4 \arctan \frac{\beta \cos(\alpha t)}{\alpha \cosh(\beta x)}; \quad \alpha^2 + \beta^2 = 1$$

For $0 < |\beta| \ll 1$ the breather disproves *asymptotic stability* of the vacuum solution: B_β is a small perturbation of 0 which does not converge to 0 for large times.

Consider the Linear Klein–Gordon equation, for V a real-valued compact perturbation of $-\partial_{xx}$

$$u_{tt} - u_{xx} + 1 + V(x) =: u_{tt} + Lu = 0 \quad (\text{LKG})$$

Linear Dynamics

Consider the Linear Klein–Gordon equation, for V a real-valued compact perturbation of $-\partial_{xx}$

$$u_{tt} - u_{xx} + 1 + V(x) =: u_{tt} + Lu = 0 \quad (\text{LKG})$$

The operator L is self adjoint and has essential spectrum $\sigma_{\text{ess}} = [1, \infty)$, and point spectrum $\sigma_d = \{\mu_j^u\}_{j=1}^N \cup \{\mu_i^s\}_{i=1}^M$ with associated eigenfunctions $\{\psi_j\}_{j=1}^N$, $\{\phi_i\}_{i=1}^M$

$$\lambda_j \in (-\infty, 0), \quad \mu_i \in [0, 1)$$

Linear Dynamics

Consider the Linear Klein–Gordon equation, for V a real-valued compact perturbation of $-\partial_{xx}$

$$u_{tt} - u_{xx} + 1 + V(x) =: u_{tt} + Lu = 0 \quad (\text{LKG})$$

The operator L is self adjoint and has essential spectrum $\sigma_{\text{ess}} = [1, \infty)$, and point spectrum $\sigma_d = \{\mu_j^u\}_{j=1}^N \cup \{\mu_i^s\}_{i=1}^M$ with associated eigenfunctions $\{\psi_j\}_{j=1}^N$, $\{\phi_i\}_{i=1}^M$

$$\lambda_j \in (-\infty, 0), \quad \mu_i \in [0, 1)$$

By superposition principle for linear equation, a solution of the evolution equation (1) is given by

$$u(x, t) = \underbrace{\sum_{j=1}^N \alpha_j e^{-\lambda_j t} \psi_j}_{\text{unstable direction}} + \underbrace{\sum_{i=1}^M (\beta_i \cos(t\sqrt{\mu_i}) + \gamma_i \sin(t\sqrt{\mu_i})) \phi_i}_{\text{oscillating internal modes}} + r(x, t)$$

The remainder r satisfies the estimate $\|r(x, t)\|_{L_x^\infty} \lesssim t^{-\frac{1}{2}}$

Small Breathers Formation

The rise of small breathers is mainly linked to two phenomena

- 1 *Linear Resonance* The linearized operator for equation (SG) $L := -\partial_{xx} + 1$ has a spectral resonance at the bottom of the continuous spectrum

$$L1 = 1, \quad 1 \in L^\infty \setminus L^2$$

¹O. Gomide, M. Guardia, T. M-Seara, C. Zeng (2025): *On small breathers of nonlinear Klein-Gordon equations via exponentially small homoclinic splitting*. *Inventiones mathematicae*

²H. Segur, M. D. Kruskal. (1987) :*Nonexistence of small-amplitude breather solutions in phi 4 theory*. Physical review letters

Small Breathers Formation

The rise of small breathers is mainly linked to two phenomena

- 1 *Linear Resonance* The linearized operator for equation (SG) $L := -\partial_{xx} + 1$ has a spectral resonance at the bottom of the continuous spectrum

$$L1 = 1, \quad 1 \in L^\infty \setminus L^2$$

A generic solution of the linear evolution has an asymptotic profile

$$u(x, t) \simeq t^{-\frac{1}{2}} (c_1 \cos(t) + c_2(\sin(t)))$$

for $c_1, c_2 \in \mathbb{R}$. The solution decays, but has a slow dispersion.

¹O. Gomide, M. Guardia, T. M-Seara, C. Zeng (2025): *On small breathers of nonlinear Klein-Gordon equations via exponentially small homoclinic splitting*. *Inventiones mathematicae*

²H. Segur, M. D. Kruskal. (1987) :*Nonexistence of small-amplitude breather solutions in phi 4 theory*. Physical review letters

Small Breathers Formation

The rise of small breathers is mainly linked to two phenomena

- 1 **Linear Resonance** The linearized operator for equation (SG) $L := -\partial_{xx} + 1$ has a spectral resonance at the bottom of the continuous spectrum

$$L1 = 1, \quad 1 \in L^\infty \setminus L^2$$

A generic solution of the linear evolution has an asymptotic profile

$$u(x, t) \simeq t^{-\frac{1}{2}} (c_1 \cos(t) + c_2(\sin(t)))$$

for $c_1, c_2 \in \mathbb{R}$. The solution decays, but has a slow dispersion.

- 2 **Nonlinear Integrability** Integrability structure leads the linear singular singular dispersion to form the nonlinear breather solution, see ¹ and ²

¹O. Gomide, M. Guardia, T. M-Seara, C. Zeng (2025): *On small breathers of nonlinear Klein-Gordon equations via exponentially small homoclinic splitting*. *Inventiones mathematicae*

²H. Segur, M. D. Kruskal. (1987) :*Nonexistence of small-amplitude breather solutions in phi 4 theory*. *Physical review letters*

Point Interaction Model

We consider the perturbation of (SG) proposed in ³, expressed in vectorial form for $u_1 := u, u_2 := \partial_t u$

$$\begin{cases} \dot{u}_1 = u_2 \\ \dot{u}_2 = (u_1)_{xx} - \sin(u) - q \sin(u) \delta_0 = 0 \end{cases} \quad (\text{SG}\delta)$$

The Dirac Delta δ_0 models a localized defect in the material centred in 0, while $q \in \mathbb{R}$ depicts strength and attractive/repulsive nature of the defect.

³Z. Fei, Y.S. Kivshar, L. Vázquez (1992) : *Resonant kink-impurity interactions in the sine-Gordon model*. Phy. Rev. Lett. A

Point Interaction Model

We consider the perturbation of (SG) proposed in ³, expressed in vectorial form for $u_1 := u, u_2 := \partial_t u$

$$\begin{cases} \dot{u}_1 = u_2 \\ \dot{u}_2 = (u_1)_{xx} - \sin(u) - q \sin(u) \delta_0 = 0 \end{cases} \quad (\text{SG}\delta)$$

The Dirac Delta δ_0 models a localized defect in the material centred in 0, while $q \in \mathbb{R}$ depicts strength and attractive/repulsive nature of the defect.

By Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow C^0(\mathbb{R})$ the Cauchy Problem (SG δ) is globally well posed for an initial datum $(u_1^0, u_2^0) \in H^1 \times L^2$ and preserves the energy

$$E(u_1, u_2) = \int \frac{(u_1)_x^2}{2} + \frac{u_2^2}{2} + 1 - \cos(u) \, dx + q(1 - \cos(u(0)))$$

³Z. Fei, Y.S. Kivshar, L. Vázquez (1992) : *Resonant kink-impurity interactions in the sine-Gordon model*. Phy. Rev. Lett. A

Main Result

Theorem (Asymptotic Stability of Vacuum)

Let $q > 0$, or $-\frac{4\sqrt{2}}{3} < q < 0$. Then there exists $\varepsilon_q > 0$ such that, if $\|(u_1^0, u_2^0)\|_{H^1 \times L^2} \leq \varepsilon_q$, then for any compact $K \subset \mathbb{R}$

$$\lim_{t \rightarrow \pm\infty} \|(u_1(t), u_2(t))\|_{H^1 \times L^2(K)} = 0$$

where (u_1, u_2) is the global solution of $(SG\delta)$, with initial datum (u_1^0, u_2^0) .

Main Result

Theorem (Asymptotic Stability of Vacuum)

Let $q > 0$, or $-\frac{4\sqrt{2}}{3} < q < 0$. Then there exists $\varepsilon_q > 0$ such that, if $\|(u_1^0, u_2^0)\|_{H^1 \times L^2} \leq \varepsilon_q$, then for any compact $K \subset \mathbb{R}$

$$\lim_{t \rightarrow \pm\infty} \|(u_1(t), u_2(t))\|_{H^1 \times L^2(K)} = 0$$

where (u_1, u_2) is the global solution of $(\text{SG}\delta)$, with initial datum (u_1^0, u_2^0) .

Corollary

For q as in the Theorem, there exist no small time-periodic solution to $(\text{SG}\delta)$.

The linearized operator for (SG δ) $L_q := -\partial_{xx} + 1 + q\delta_0$ is self adjoint over its natural domain

$$\mathcal{D}(L_q) := \left\{ v \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} \setminus \{0\}) \mid v_x(0^+) - v_x(0^-) = qv(0) \right\}$$

and has essential spectrum $\sigma_{\text{ess}} = [1, \infty)$.

The linearized operator for (SG δ) $L_q := -\partial_{xx} + 1 + q\delta_0$ is self adjoint over its natural domain

$$\mathcal{D}(L_q) := \left\{ v \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} \setminus \{0\}) \mid v_x(0^+) - v_x(0^-) = qv(0) \right\}$$

and has essential spectrum $\sigma_{\text{ess}} = [1, \infty)$.

For $q > 0$ there exists no point spectrum and the resonance interaction is broken.

The linearized operator for (SG δ) $L_q := -\partial_{xx} + 1 + q\delta_0$ is self adjoint over its natural domain

$$\mathcal{D}(L_q) := \left\{ v \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} \setminus \{0\}) \mid v_x(0^+) - v_x(0^-) = qv(0) \right\}$$

and has essential spectrum $\sigma_{\text{ess}} = [1, \infty)$.

For $q > 0$ there exists no point spectrum and the resonance interaction is broken.

For $-2 < q < 0$, there is again no resonance interaction but the point spectrum contains an internal mode with explicit eigenfunction

$$\mu_q = 1 - \frac{q^2}{4} \in (0, 1), \quad H_q := e^{\frac{q|x|}{2}}$$

The linearized operator for $(SG\delta)$ $L_q := -\partial_{xx} + 1 + q\delta_0$ is self adjoint over its natural domain

$$\mathcal{D}(L_q) := \left\{ v \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} \setminus \{0\}) \mid v_x(0^+) - v_x(0^-) = qv(0) \right\}$$

and has essential spectrum $\sigma_{ess} = [1, \infty)$.

For $q > 0$ there exists no point spectrum and the resonance interaction is broken.

For $-2 < q < 0$, there is again no resonance interaction but the point spectrum contains an internal mode with explicit eigenfunction

$$\mu_q = 1 - \frac{q^2}{4} \in (0, 1), \quad H_q := e^{\frac{q|x|}{2}}$$

In this case, asymptotic stability is recovered by the Nonlinear Fermi Golden Rule: under proper assumption on q , the nonlinearity has a coupling effect between the oscillating internal mode and the dispersive component.

The setting of the problem is in low dimension, with low power nonlinearities. Dispersive estimates are hard to implement.

⁴M. Kowalczyk, Y. Martel, C. Muñoz. (2017) :*Nonexistence of small, odd breathers for a class of nonlinear wave equations*. Letters in Mathematical Physics

Sketches of proof

The setting of the problem is in low dimension, with low power nonlinearities. Dispersive estimates are hard to implement.

For a solution u , we define following ⁴ a virial functional $\mathcal{I}_u$ such that

$$\mathcal{I}_u(t) \lesssim \left\| \begin{pmatrix} u_1^0 \\ u_2^0 \end{pmatrix} \right\|_{H^1 \times L^2} ; \quad \frac{d}{dt} \mathcal{I}_u \gtrsim \|(u_1(t), u_2(t))\|_K^2$$

where $\|\cdot\|_K$ is a proper norm in the compact $K \subset \mathbb{R}$.

⁴M. Kowalczyk, Y. Martel, C. Muñoz. (2017) : *Nonexistence of small, odd breathers for a class of nonlinear wave equations*. Letters in Mathematical Physics

Sketches of proof

The setting of the problem is in low dimension, with low power nonlinearities. Dispersive estimates are hard to implement.

For a solution u , we define following ⁴ a virial functional $\mathcal{I}_u$ such that

$$\mathcal{I}_u(t) \lesssim \left\| (u_1^0, u_2^0) \right\|_{H^1 \times L^2} ; \quad \frac{d}{dt} \mathcal{I}_u \gtrsim \left\| (u_1(t), u_2(t)) \right\|_K^2$$

where $\| \cdot \|_K$ is a proper norm in the compact $K \subset \mathbb{R}$.

Integrating the inequality we get

$$\int_0^\infty \left\| (u_1(t), u_2(t)) \right\|_K^2 dt \lesssim \left\| (u_1^0, u_2^0) \right\|_{H^1 \times L^2}$$

and hence the thesis.

⁴M. Kowalczyk, Y. Martel, C. Muñoz. (2017) : *Nonexistence of small, odd breathers for a class of nonlinear wave equations*. Letters in Mathematical Physics

Case $q > 0$

Let (u_1, u_2) be a solution of $(SG\delta)$. For $\psi \in C^\infty \cap L^\infty$, with $\psi' > 0$, $\psi(0) = 0$ we define

$$\mathcal{I}_u(t) = \int_{\mathbb{R}} \left(\psi \partial_x u + \frac{1}{2} \psi' u \right) \partial_t u$$

We can compute explicitly

$$\begin{aligned} -\frac{d}{dt} \mathcal{I} &= - \int_{\mathbb{R}} \left(\psi \partial_x u_2 + \frac{1}{2} \psi' u_2 \right) u_2 - \int_{\mathbb{R}} \left(\psi \partial_x u_1 + \frac{1}{2} \psi' u_1 \right) \partial_{tt} u_1 \\ &= \int w_x^2 + \frac{1}{2} \int \left(\frac{\zeta''}{\zeta} - \frac{(\zeta')^2}{\zeta^2} \right) w^2 + \frac{q}{2} w^2(0) + \mathcal{O}(u_1) =: \mathcal{B}_q(w, w) + \mathcal{O}(u_1) \end{aligned}$$

where

$$w = \zeta u; \quad \zeta = \sqrt{\psi'}; \quad \mathcal{O}(u) \leq C \|u_1^2 \zeta\|_{L^2}^2 + u_1(0)^4$$

Case $q > 0$

$$-\frac{d}{dt}\mathcal{I} = \mathcal{B}_q(w, w) + \mathcal{O}(u_1), \quad \mathcal{B}_q(w, w) = \int w_x^2 + \underbrace{\int \frac{1}{2} \left(\frac{\zeta''}{\zeta} - \frac{(\zeta')^2}{\zeta^2} \right) w^2}_{:=V(x)} + \frac{q}{2} w^2(0) + \mathcal{O}(u_1)$$

⁵M. Kowalczyk, Y. Martel, C. Muñoz. (2017) : *Nonexistence of small, odd breathers for a class of nonlinear wave equations*. Letters in Mathematical Physics

Case $q > 0$

$$-\frac{d}{dt}\mathcal{I} = \mathcal{B}_q(w, w) + \mathcal{O}(u_1), \quad \mathcal{B}_q(w, w) = \int w_x^2 + \underbrace{\int \frac{1}{2} \left(\frac{\zeta''}{\zeta} - \frac{(\zeta')^2}{\zeta^2} \right) w^2}_{:=V(x)} + \frac{q}{2} w^2(0) + \mathcal{O}(u_1)$$

In ⁵ for $q = 0$ the auxiliary function are chosen so that $V = \frac{1}{2\lambda^2} \operatorname{sech}^2\left(\frac{x}{\lambda}\right)$. The linear operator associated to the bilinear form $\mathcal{B}_0$ has well known spectrum property; specifically

$$\mathcal{B}_0(w, w) \geq \frac{1}{4} \int w_x^2 \quad \text{for any odd } w \in H^1$$

⁵M. Kowalczyk, Y. Martel, C. Muñoz. (2017) : *Nonexistence of small, odd breathers for a class of nonlinear wave equations*. Letters in Mathematical Physics

Case $q > 0$

$$-\frac{d}{dt}\mathcal{I} = \mathcal{B}_q(w, w) + \mathcal{O}(u_1), \quad \mathcal{B}_q(w, w) = \int w_x^2 + \underbrace{\int \frac{1}{2} \left(\frac{\zeta''}{\zeta} - \frac{(\zeta')^2}{\zeta^2} \right) w^2}_{:=V(x)} + \frac{q}{2} w^2(0) + \mathcal{O}(u_1)$$

In ⁵ for $q = 0$ the auxiliary function are chosen so that $V = \frac{1}{2\lambda^2} \operatorname{sech}^2\left(\frac{x}{\lambda}\right)$. The linear operator associated to the bilinear form $\mathcal{B}_0$ has well known spectrum property; specifically

$$\mathcal{B}_0(w, w) \geq \frac{1}{4} \int w_x^2 \quad \text{for any odd } w \in H^1$$

For $q > 0$, the additional positive term in $\mathcal{B}_q$ implies the coercivity estimate for any $w \in H^1$, for λ sufficiently large.

⁵M. Kowalczyk, Y. Martel, C. Muñoz. (2017) : *Nonexistence of small, odd breathers for a class of nonlinear wave equations*. Letters in Mathematical Physics

Case $q > 0$

$$-\frac{d}{dt}\mathcal{I} = \mathcal{B}_q(w, w) + \mathcal{O}(u_1), \quad \mathcal{B}_q(w, w) = \int w_x^2 + \underbrace{\int \frac{1}{2} \left(\frac{\zeta''}{\zeta} - \frac{(\zeta')^2}{\zeta^2} \right) w^2}_{:=V(x)} + \frac{q}{2} w^2(0) + \mathcal{O}(u_1)$$

In ⁵ for $q = 0$ the auxiliary function are chosen so that $V = \frac{1}{2\lambda^2} \operatorname{sech}^2\left(\frac{x}{\lambda}\right)$. The linear operator associated to the bilinear form $\mathcal{B}_0$ has well known spectrum property; specifically

$$\mathcal{B}_0(w, w) \geq \frac{1}{4} \int w_x^2 \quad \text{for any odd } w \in H^1$$

For $q > 0$, the additional positive term in $\mathcal{B}_q$ implies the coercivity estimate for any $w \in H^1$, for λ sufficiently large. Nonlinear terms are higher order perturbations, so that in terms of u_1

$$-\frac{d}{dt}\mathcal{I} \gtrsim_{q,\lambda} \|\zeta u_1\|_{L^2}^2 + \|\zeta \partial_x u_1\|_{L^2}^2$$

⁵M. Kowalczyk, Y. Martel, C. Muñoz. (2017) : *Nonexistence of small, odd breathers for a class of nonlinear wave equations*. Letters in Mathematical Physics

Case $q > 0$

$$-\frac{d}{dt}\mathcal{I} = \mathcal{B}_q(w, w) + \mathcal{O}(u_1), \quad \mathcal{B}_q(w, w) = \int w_x^2 + \underbrace{\int \frac{1}{2} \left(\frac{\zeta''}{\zeta} - \frac{(\zeta')^2}{\zeta^2} \right) w^2}_{:=V(x)} + \frac{q}{2} w^2(0) + \mathcal{O}(u_1)$$

In ⁵ for $q = 0$ the auxiliary function are chosen so that $V = \frac{1}{2\lambda^2} \operatorname{sech}^2\left(\frac{x}{\lambda}\right)$. The linear operator associated to the bilinear form $\mathcal{B}_0$ has well known spectrum property; specifically

$$\mathcal{B}_0(w, w) \geq \frac{1}{4} \int w_x^2 \quad \text{for any odd } w \in H^1$$

For $q > 0$, the additional positive term in $\mathcal{B}_q$ implies the coercivity estimate for any $w \in H^1$, for λ sufficiently large. Nonlinear terms are higher order perturbations, so that in terms of u_1

$$-\frac{d}{dt}\mathcal{I} \gtrsim_{q,\lambda} \|\zeta u_1\|_{L^2}^2 + \|\zeta \partial_x u_1\|_{L^2}^2$$

which can be completed with u_2 to infer

$$\int_0^\infty \|\zeta u_1\|_{L^2}^2 + \|\zeta \partial_x u_1\|_{L^2}^2 + \|\zeta u_2\|_{L^2}^2 dt \lesssim \left\| (u_1^0, u_2^0) \right\|_{H^1 \times L^2}^2$$

⁵M. Kowalczyk, Y. Martel, C. Muñoz. (2017) : *Nonexistence of small, odd breathers for a class of nonlinear wave equations*. Letters in Mathematical Physics

Case $q < 0$

We split the solution as $u_1 = z_1 + aH_q$, $u_2 = z_2 + bH_q$, so that $(z_1, H_q)_{L^2} = (z_2, H_q)_{L^2} = 0$.
The evolution equation satisfy

$$\begin{cases} \dot{a} = \mu b \\ \dot{b} = -\mu b + ca^3 + \mathcal{R}_1 \end{cases}$$

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = -L_q z_1 - \frac{a^3 H_q^3}{3!} + \mathcal{R}_2 \end{cases}$$

Case $q < 0$

We split the solution as $u_1 = z_1 + aH_q$, $u_2 = z_2 + bH_q$, so that $(z_1, H_q)_{L^2} = (z_2, H_q)_{L^2} = 0$. The evolution equation satisfy

$$\begin{cases} \dot{\alpha} = 3\sqrt{\mu}\beta + \tilde{\mathcal{R}}_\alpha \\ \dot{\beta} = -3\sqrt{\mu}\alpha + \tilde{\mathcal{R}}_\beta \end{cases}$$

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = -L_q z_1 - \frac{\alpha H_q^3}{3!} + \tilde{\mathcal{R}}_2 \end{cases}$$

for α, β normal forms, cubic expression of a, b .

Case $q < 0$

We split the solution as $u_1 = z_1 + aH_q$, $u_2 = z_2 + bH_q$, so that $(z_1, H_q)_{L^2} = (z_2, H_q)_{L^2} = 0$. The evolution equation satisfy

$$\begin{cases} \dot{\alpha} = 3\sqrt{\mu}\beta + \tilde{\mathcal{R}}_\alpha \\ \dot{\beta} = -3\sqrt{\mu}\alpha + \tilde{\mathcal{R}}_\beta \end{cases} \quad \begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = -L_q z_1 - \frac{\alpha H_q^3}{3!} + \tilde{\mathcal{R}}_2 \end{cases}$$

for α, β normal forms, cubic expression of a, b .

We can repeat the virial computation as above for $\tilde{\mathcal{I}}_1 = \int (\psi(z_1)_x + \frac{1}{2}\psi' z_1) z_2$ and infer

$$\|\zeta v_1\|_{L^2}^2 + \|\zeta \partial_x v_1\|_{L^2}^2 \lesssim -\frac{d}{dt} \tilde{\mathcal{I}} + |a|^6 + |b|^6$$

Case $q < 0$

The Nonlinear Fermi Golden condition holds if there exists a $g \in L^\infty$ such that

$$L_q g = 9\mu g; \quad \int H_q^3 g \, dx = \Gamma \neq 0 \quad (\text{NLFG})$$

Case $q < 0$

The Nonlinear Fermi Golden condition holds if there exists a $g \in L^\infty$ such that

$$L_q g = 9\mu g; \quad \int H_q^3 g \, dx = \Gamma \neq 0 \quad (\text{NLFG})$$

We can compute the derivative of a new virial quantity

$$\begin{aligned} \frac{d}{dt} \underbrace{\left(-\alpha \int z_2 g \psi + 3\sqrt{\mu}\beta \int z_1 g \psi \right)}_{:=\mathcal{J}} &= -\alpha \int (L_q + 9\mu) z_1 g \psi + \alpha^2 \int g \psi H_q^3 + \tilde{\mathcal{R}}_1 \\ &= \alpha^2 \Gamma + \tilde{\mathcal{R}}_2 \end{aligned}$$

with $|\tilde{\mathcal{R}}_2| \ll \|z_1 \zeta\|_{L^2}^2$.

Case $q < 0$

The Nonlinear Fermi Golden condition holds if there exists a $g \in L^\infty$ such that

$$L_q g = 9\mu g; \quad \int H_q^3 g \, dx = \Gamma \neq 0 \quad (\text{NLFG})$$

We can compute the derivative of a new virial quantity

$$\begin{aligned} \frac{d}{dt} \underbrace{\left(-\alpha \int z_2 g \psi + 3\sqrt{\mu}\beta \int z_1 g \psi \right)}_{:=\mathcal{J}} &= -\alpha \int (L_q + 9\mu) z_1 g \psi + \alpha^2 \int g \psi H_q^3 + \tilde{\mathcal{R}}_1 \\ &= \alpha^2 \Gamma + \tilde{\mathcal{R}}_2 \end{aligned}$$

with $|\tilde{\mathcal{R}}_2| \ll \|z_1 \zeta\|_{L^2}^2$.

A control over $|\alpha|^2$ allows to recover a bound over $|a|^6 + |b|^6$ and hence

$$\int_0^\infty |a|^6 + |b|^6 + \|(z_1, z_2)\|_{H^1 \times L^2(K)}^2 \, dt \lesssim \|(u_1^0, u_2^0)\|_{H^1 \times L^2}^2$$

Thank you for the attention

Symmetry-type conditions on conditional bases of Banach spaces

Alejandro Marcos Baños

 almarcb@unirioja.es

Universidad de la Rioja

 Encuentros de Análisis Real y Complejo

Logroño, 30/05/2026

A. Marcos acknowledges the support from the University of La Rioja under an FPI predoctoral contract.



**UNIVERSIDAD
DE LA RIOJA**

1 Introduction

2 Renormings through a semigroup of operators

3 Renorming spreading and symmetric systems

4 Connections with greedy-like properties



**UNIVERSIDAD
DE LA RIOJA**

Definition - Biorthogonal systems and minimal systems

A **biorthogonal system** of a quasi-Banach space $\mathbb{X}$ is a sequence of pairs $(x_n, x_n^*)_{n=1}^\infty$ in $\mathbb{X} \times \mathbb{X}^*$ such that

$$x_n^*(x_k) = \delta_{n,k} := \begin{cases} 1, & \text{if } n = k, \\ 0, & \text{otherwise.} \end{cases}$$

A sequence $(x_n)_{n=1}^\infty$ for which such $(x_n^*)_{n=1}^\infty$ exists is called a **minimal system**.

Definition - Fundamental systems

A minimal system $(x_n)_{n=1}^{\infty}$ is said to be **fundamental** if

$$\overline{\text{span}}(x_n : n \in \mathbb{N}) = \mathbb{X}.$$

Definition - Totality

A minimal system $(x_n)_{n=1}^{\infty}$ is said to be **total** if the **coordinate transform**

$$\begin{aligned}\mathbb{X} &\rightarrow \mathbb{F}^{\mathbb{N}} \\ x &\mapsto (x_n^*(x))_{n=1}^{\infty}\end{aligned}$$

is one-to-one.

Definition - Markushevich bases

A minimal system is a **Markushevich basis** if it is both total and complete.

Definition - Schauder bases

Let $\mathbb{X}$ be a quasi-Banach space. A sequence $(x_n)_{n=1}^{\infty}$ is a **Schauder basis** for $\mathbb{X}$ if for every element $x \in \mathbb{X}$ there exists a unique sequence $(\lambda_n)_{n=1}^{\infty}$ such that

$$x = \sum_{n=1}^{\infty} \lambda_n x_n.$$

Definition - Schauder bases

Let $\mathbb{X}$ be a quasi-Banach space. A sequence $(x_n)_{n=1}^{\infty}$ is a **Schauder basis** for $\mathbb{X}$ if for every element $x \in \mathbb{X}$ there exists a unique sequence $(\lambda_n)_{n=1}^{\infty}$ such that

$$x = \sum_{n=1}^{\infty} \lambda_n x_n.$$

Every Schauder basis is also a Markushevich basis. The coordinate functionals are the maps

$$x_n^*(x) = \lambda_n, \quad x \in \mathbb{X}.$$

Example - $L_p(\mathbb{T})$

The trigonometric system associated with the Fourier series

$$(e^{2\pi i n t})_{n=-\infty}^{\infty}$$

with the arrangement $\{0, 1, -1, 2, -2, \dots\}$ is a Schauder basis for $L_p(\mathbb{T})$ whenever $1 < p < \infty$.

Definition - Unconditional bases

Let $\mathbb{X}$ be a quasi-Banach space. A sequence $(x_n)_{n=1}^{\infty}$ is an **unconditional basis** for $\mathbb{X}$ if for every element $x \in \mathbb{X}$ there exists a unique sequence $(\lambda_n)_{n=1}^{\infty}$ such that the series

$$\sum_{n=1}^{\infty} \lambda_n x_n$$

converges to x unconditionally.

Definition - Unconditional bases

Let $\mathbb{X}$ be a quasi-Banach space. A sequence $(x_n)_{n=1}^{\infty}$ is an **unconditional basis** for $\mathbb{X}$ if for every element $x \in \mathbb{X}$ there exists a unique sequence $(\lambda_n)_{n=1}^{\infty}$ such that the series

$$\sum_{n=1}^{\infty} \lambda_n x_n$$

converges to x unconditionally.

Every unconditional basis is a Schauder basis for every ordering of $\mathbb{N}$.

Example - $L_p([0, 1])$

The Haar system $(h_n(t))_{n=0}^{\infty}$ defined by $h_0 = \chi_{[0,1]}$ and

$$h_n(t) = \chi_{\left[\frac{2j-2}{2^{k+1}}, \frac{2j-1}{2^{k+1}}\right)}(t) - \chi_{\left[\frac{2j-1}{2^{k+1}}, \frac{2j}{2^{k+1}}\right)}(t), \quad n \in \mathbb{N},$$

where $n = 2^k + j$, as an unconditional basis of $L_p([0, 1])$ for $1 < p < \infty$.

Definition - Equivalent systems

Let $\mathbb{X}$ and $\mathbb{Y}$ be quasi-Banach spaces. Two fundamental systems $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ of $\mathbb{X}$ and $\mathbb{Y}$ respectively are **equivalent** if there exists an isomorphism $T: \mathbb{X} \rightarrow \mathbb{Y}$ such that $T(x_n) = y_n$ for all $n \in \mathbb{N}$.

Definition - Symmetric bases

Let $\mathbb{X}$ be a quasi-Banach space. A fundamental minimal system $(x_n)_{n=1}^{\infty}$ is **symmetric** if it is equivalent to $(x_{\pi(n)})_{n=1}^{\infty}$ for every bijection $\pi: \mathbb{N} \rightarrow \mathbb{N}$.

Definition - Symmetric bases

Let $\mathbb{X}$ be a quasi-Banach space. A fundamental minimal system $(x_n)_{n=1}^{\infty}$ is **symmetric** if it is equivalent to $(x_{\pi(n)})_{n=1}^{\infty}$ for every bijection $\pi: \mathbb{N} \rightarrow \mathbb{N}$.

Every Schauder symmetric basis is unconditional, but there are Banach spaces with symmetric fundamental systems that fail to be total.

Example - ℓ_p

The unit vector system $(e_n)_{n=1}^\infty$ defined by

$$e_n = (\delta_{k,n})_{k=1}^\infty, \quad k \in \mathbb{N}$$

is a symmetric basis of ℓ_p for any $0 < p < \infty$, and of c_0 .

Example (Albiac, Ansorena, Blasco, Chu, Oikhberg; 2024)

Let $\omega = (\omega_j)_{j=1}^{\infty}$ be a weight with $\lim_{j \rightarrow \infty} \omega_j > 0$ and $f = (a_n)_{n=1}^{\infty}$. Consider the norm

$$\|f\| = \sup_{\pi \text{ bijective}} \left| \sum_{j=1}^{\infty} a_{\pi(j)} \omega_j \right|.$$

The unit vector system is a symmetric conditional minimal system of the completion of the space c_{00} under the norm $\|\cdot\|$.

Definition - Spreading bases

Let $\mathbb{X}$ be a quasi-Banach space. A fundamental minimal system $(x_n)_{n=1}^{\infty}$ is **spreading** if it is equivalent to every subsequence.

Definition - Subsymmetric bases

Let $\mathbb{X}$ be a quasi-Banach space. A fundamental minimal system $(x_n)_{n=1}^{\infty}$ is **subsymmetric** if it is unconditional and spreading.

Example - c_0

The summing basis $(s_n)_{n=1}^{\infty}$ defined by

$$s_n = \sum_{j=1}^n e_j = (\underbrace{1, \dots, 1}_{n \text{ times}}, 0, \dots),$$

is a spreading conditional basis of c_0 .

Looking for isometric renormings

A classical problem is to determine whether quasi-Banach spaces can be renormed so that the above conditions hold isometrically. The standard method to address this task is:

- proving that these properties can be characterized in terms of the uniform boundedness of suitable families of operators, and
- using that this family is a semigroup.

1 Introduction

2 Renormings through a semigroup of operators

3 Renorming spreading and symmetric systems

4 Connections with greedy-like properties



**UNIVERSIDAD
DE LA RIOJA**

Lemma

Let $\mathbb{X}$ be a quasi-Banach space. If $\mathcal{G}$ is a uniformly bounded semigroup of endomorphisms of $\mathbb{X}$, the map

$$f \mapsto |||f||| = \sup_{T \in \mathcal{G}} \|T(f)\|$$

defines an equivalent quasi-norm for $\mathbb{X}$ relative to which $\|T\| \leq 1$ for all $T \in \mathcal{G}$.

Definition - Projections

Let $(x_n)_{n=1}^{\infty}$ be a fundamental minimal system and $A \subseteq \mathbb{N}$. We define the **projection operator on A** as

$$S_A: \mathbb{X} \rightarrow \mathbb{X}$$
$$x \mapsto \sum_{n \in A} x_n^*(x) x_n.$$

Definition - Shifts

Let $(x_n)_{n=1}^{\infty}$ be a fundamental minimal system and $\varphi: \mathbb{N} \rightarrow \mathbb{N}$ an injective map. We define the **shifting operator associated with φ** as

$$T_{\varphi}: \mathbb{X} \rightarrow \mathbb{X}$$
$$x \mapsto \sum_{n=1}^{\infty} x_n^*(x) x_{\varphi(n)}.$$

Semigroup for Schauder bases

$$\{S_{\{1,\dots,n\}} : n \in \mathbb{N}\}$$

Semigroup for Schauder bases

$$\{S_{\{1,\dots,n\}} : n \in \mathbb{N}\}$$

Semigroup for unconditional bases

$$\{S_A : A \subseteq \mathbb{N} \text{ finite}\}$$

Semigroup for Schauder bases

$$\{S_{\{1,\dots,n\}}: n \in \mathbb{N}\}$$

Semigroup for unconditional bases

$$\{S_A: A \subseteq \mathbb{N} \text{ finite}\}$$

Semigroup for symmetric Schauder bases

$$\{T_\pi: \pi: \mathbb{N} \rightarrow \mathbb{N} \text{ bijective}\}$$

Semigroup for subsymmetric bases

The straightforward family

$$\{T_\varphi: \varphi: \mathbb{N} \rightarrow \mathbb{N} \text{ increasing}\}$$

does not work.

Semigroup for subsymmetric bases (Ansorena; 2017)

$$\bigcup_{\substack{A \subset \mathbb{N} \\ |A| < \infty}} \{T_\varphi : \varphi : A \rightarrow \mathbb{N} \text{ increasing}\}$$

Question

Can we find an isometric renorming for spreading and symmetric non-Schauder fundamental minimal systems?

1 Introduction

2 Renormings through a semigroup of operators

3 Renorming spreading and symmetric systems

4 Connections with greedy-like properties



UNIVERSIDAD
DE LA RIOJA

Lemma

If $(x_n)_{n=1}^{\infty}$ is a symmetric fundamental minimal system of a quasi-Banach space, then

$$\sup \{ \|T_{\pi}\| : \pi \text{ bijection on } \mathbb{N} \} < \infty.$$

Lemma

If $(x_n)_{n=1}^{\infty}$ is a symmetric fundamental minimal system of a quasi-Banach space, then

$$\sup \{ \|T_{\pi}\| : \pi \text{ bijection on } \mathbb{N} \} < \infty.$$

Lemma

If $(x_n)_{n=1}^{\infty}$ is a spreading fundamental minimal system of a quasi-Banach space, then

$$\sup \{ \|T_{\psi}\| : \psi : A \rightarrow \mathbb{N} \text{ increasing, } A \in \mathcal{P}_{<\infty}(\mathbb{N}) \} < \infty.$$

Theorem

Every symmetric fundamental minimal system admits an isometric renorming.

$$\|f\| = \sup_{T \in \mathcal{G}_m} \|T_\pi\| = \sup \{ \|T_\pi\| : \pi \text{ bijection on } \mathbb{N} \}.$$

We have not found a suitable semigroup for renorming spreading systems.

Spreading models

A sequence $\mathcal{X} = (x_n)_{n=1}^{\infty}$ in a quasi-Banach space $\mathbb{X}$ is said to be **good** if the following limit exists

$$\|\cdot\|_{\mathcal{X}} : c_{00} \longrightarrow [0, \infty)$$
$$(a_n)_{n=1}^{\infty} \longmapsto \lim_{\varphi \in [\mathbb{N}]^{(N)}} \left\| \sum_{j=1}^N a_j x_{\varphi(j)} \right\|,$$

where $N = \max \text{supp}((a_n)_{n=1}^{\infty})$ and $[\mathbb{N}]^{(N)} = \{\varphi : \mathbb{N} \cap [1, N] \rightarrow \mathbb{N} : \varphi \text{ increasing}\}$ is a directed set endowed with the partial order $\varphi \leq \psi$ if $\varphi(1) \leq \psi(1)$.

Every good sequence is bounded and, by the Infinite Ramsey Theorem, every bounded sequence contains a good subsequence, so $\|\cdot\|_{\mathcal{X}}$ is a well-defined quasi-norm.

Spreading models

The completion of the space $(c_{00}, \|\cdot\|_{\mathcal{X}})$ is the **spreading model associated with $\mathcal{X}$** and will be denoted by $\mathbb{S}[\mathcal{X}]$. The unit vector system is an isometrically spreading fundamental minimal system of $\mathbb{S}[\mathcal{X}]$.

Theorem (Ansorena, M.; 2026)

Let $\mathcal{X} = (x_n)_{n=1}^{\infty}$ be a spreading fundamental minimal system of a quasi-Banach space $\mathbb{X}$. There exists an equivalent quasi-norm that makes $\mathcal{X}$ isometrically spreading.

Theorem (Ansorena, M.; 2026)

Let $\mathcal{X} = (x_n)_{n=1}^{\infty}$ be a spreading fundamental minimal system of a quasi-Banach space $\mathbb{X}$. There exists an equivalent quasi-norm that makes $\mathcal{X}$ isometrically spreading.

Sketch of the proof

The system $\mathcal{X}$ is equivalent to all of its subsequences, so we can assume that it is good. By the uniform boundedness of the shift operators, $\mathcal{X}$ is equivalent to the unit vector system of $\mathbb{S}[\mathcal{X}]$. Since this system is isometrically spreading, we are done.

1 Introduction

2 Renormings through a semigroup of operators

3 Renorming spreading and symmetric systems

4 Connections with greedy-like properties



**UNIVERSIDAD
DE LA RIOJA**

Connections between symmetry and greedy-like properties



Theorem (Albiac, Ansorena, Blasco, Chu, Oikhberg; 2024)

Let $\mathcal{X}$ be a symmetric fundamental minimal system of a Banach space. The following are equivalent.

- The system $\mathcal{X}$ is unconditional.
- The system $\mathcal{X}$ is bidemocratic.
- The system $\mathcal{X}$ is unconditional for constant coefficients.
- The system $\mathcal{X}$ is a Markushevich basis.

Connections between symmetry and greedy-like properties



Theorem (Albiac, Ansorena, Berasategui, Berná; 2025)

There exists a **quasi-Banach space** with a symmetric almost greedy Markushevich basis that fails to be a Schauder basis under any rearrangement.

Connections between symmetry and greedy-like properties



Theorem (Ansorena, Berasategui, M.; 2026→)

Let $\mathcal{X}$ be a spreading fundamental minimal system of a Banach space. The following are equivalent.

- The system $\mathcal{X}$ is unconditional.
- The system $\mathcal{X}$ is bidemocratic.
- The system $\mathcal{X}$ is unconditional for constant coefficients.

Symmetry-type conditions on conditional bases of Banach spaces

Alejandro Marcos Baños

 almarcb@unirioja.es



UNIVERSIDAD
DE LA RIOJA